

# Symmetry and Asymmetry for $n$ th-degree Algebraic Functions and the Tangent Lines

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**Abstract:** We reveal one relationship between each degree algebraic function and its tangent line, via its derivative. In particular, it is easy to see and well known that asymmetry (resp. symmetry) of tangent lines of a quadratic (resp. cubic) function at its minimum and maximum zero points, but it is not easy to investigate symmetry and asymmetry of them of  $n$ th-degree functions if  $n$  is 4 or more. We thus investigate the relationship between the slopes of the tangent lines at minimum and maximum zero points of the  $n$ th-degree function. We will in this note be able to know some sufficient conditions for the ratio of their slopes to be 1 or -1. By these, we can understand that tangent lines at minimum and maximum zero points have a symmetrical (resp. asymmetrical) relationship if the ratio of their slopes is -1 (resp. 1). In other words, these properties give us symmetry and asymmetry of the functions. Furthermore, we also mention the property of the discriminant of a quadratic function.

**Keywords:** Algebraic Equation, Algebraic Function, Zero Point, Tangent Line, Discriminant, Inflection Point

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## 1. Introduction

It is important to focus on symmetry in mathematics. Symmetry simplifies solving problems and reveals the qualitative aspects of mathematical problems and / or mathematical objects. For example, symmetry is effective for problems on the graphs of functions (See e.g. [12] for various graphs of explicit and implicit functions). We thus consider symmetry and asymmetry for  $n$ th-degree algebraic functions and those tangent lines (See e.g. [4] for the history of tangents of functions) in this note.

In Section 2, we first check asymmetry for the tangent lines of a quadratic function ( $n = 2$ ). We also mention the property of the discriminant of a quadratic function in that section.

In Section 3, we next check symmetry for the tangent lines of a cubic function ( $n = 3$ ). It will be known again, in that section, that cubic functions have symmetry at the inflection points and that property is useful when considering cubic functions.

In Section 4, we finally check symmetry and asymmetry for the tangent lines of an  $n$ th-degree algebraic function  $h$ . It will be known, in that section, that symmetry and asymmetry of the

tangent lines of  $h$  at the minimum and maximum zero points of  $h$  generally depend on the degree  $n$  and the position of the zero points. Here, zero points of an  $n$ th-degree algebraic equation are the real solutions of it.

## 2. Quadratic Equations, Functions and the Tangent Lines

We consider a quadratic equation

$$ax^2 + bx + c = 0 \quad (1)$$

where  $a \in \mathbb{R} \setminus \{0\}$  and  $b, c \in \mathbb{R}$ . We suppose that (1) has two different real solutions  $\alpha, \beta$  such that  $\alpha < \beta$ . It is well known that the solutions (roots) of (1) is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and this is called the quadratic formula. The above two roots are real numbers if  $b^2 - 4ac > 0$ ; are equal real numbers if

$b^2 - 4ac = 0$ ; are imaginary numbers if  $b^2 - 4ac < 0$ . In this way,  $D := b^2 - 4ac$  is a judgment value that separates kinds of the roots, so  $D$  is called the discriminant of (1).

We put  $f(x) = ax^2 + bx + c$ . Then,  $f$  is called the quadratic function and it corresponds the parabola whose vertex is  $(-b/2a, -(b^2 - 4ac)/4a)$ :

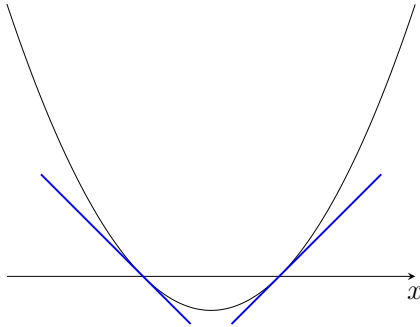
$$f(x) = a \left( x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}.$$

(Thus, remark that the discriminant shows how far the function is from the origin.) It is also well known that the coefficients  $a$ ,  $b$  and  $c$  represent the ‘ratio’ of the parabola, the slope of tangent line at the  $y$ -intercept and the  $y$ -intercept respectively. Throughout this note, we denote the first-order (resp. second-order) derivative ([5, 8, 13] etc.) of a differentiable function  $f$  of arbitrary degree by  $f'$  (resp.  $f''$ ):

$$f' = \frac{df}{dx}.$$

**Proposition 2.1.** The slopes of two tangent lines at zero points  $\alpha, \beta$  ( $\alpha \neq \beta$ ) of  $f$  are different signs, that is,

$$\frac{f'(\alpha)}{f'(\beta)} = -1.$$



**Figure 1.** Quadratic function and the tangent lines.

*proof.* It is obvious, since the parabola  $f$  is symmetric with respect to the axis  $x = -b/2a$ . (Also, refer to Theorem 4.1 later.)

We prove that the discriminants of quadratic equations can be represented by those derivatives. Recall that the discriminant of (1) is given by  $b^2 - 4ac$ .

**Proposition 2.2.** Let  $D = b^2 - 4ac$  be a discriminant of (1). Then, one has

$$D = \{f'(\alpha)\}^2 = \{f'(\beta)\}^2.$$

*proof.* Let the roots  $\alpha$  and  $\beta$  be represented by  $\sigma$ :  $\sigma = \alpha$  or  $\sigma = \beta$ . We multiply both sides of

$$a\sigma^2 + b\sigma + c = 0$$

by  $4a$ . Then,

$$0 = 4a^2\sigma^2 + 4ab\sigma + 4ac = (2a\sigma + b)^2 - b^2 + 4ac,$$

so we have

$$D = b^2 - 4ac = (2a\sigma + b)^2 = \{f'(\sigma)\}^2.$$

This completes the proof.

**Remark 2.1.** i) It is known [14] that the discriminant of any degree algebraic equation in general expresses by using all roots of it. To be specific, if  $\alpha_1, \dots, \alpha_n$  are roots of the  $n$ th-degree algebraic equation

$$f(x) = a \prod_{j=1}^n (x - \alpha_j) = 0 \quad (2)$$

where  $a \neq 0$ , the discriminant  $D$  of  $f$  can be written as

$$D = (-1)^{(n-1)n/2} a^{n-2} \prod_{j=1}^n f'(\alpha_j).$$

ii) It is also known [14] that if we write  $M$  for the bound on the absolute value of any root  $\alpha_j$  of (2), one has

$$|\alpha_j - \alpha_k| \geq \frac{|\sqrt{D}|}{(2M)^{(n-1)n/2-1}}$$

for any  $j \neq k$ . Here, it goes without saying that  $D$  stands for the discriminant of  $f$ .

iii) We can see deeper consideration for discriminants in e.g. [3].

**Example 2.1.** Proposition 2.1 can be applied to the following problem:

‘Suppose that the quadratic equation  $3x^2 + px + 1 = 0$  has a root  $-1$ . Then, find the value of  $p$  and the other root.’

The solution is as follows:

Since  $-1$  is a root of the given quadratic equation,

$$3(-1)^2 + p(-1) + 1 = 0 \quad \text{i.e.} \quad p = 4.$$

Also, the derivative  $f'(x) = 6x + p$  of  $f(x) := 3x^2 + px + 1$  implies

$$6\alpha + 4 = -\{6(-1) + 4\} \quad \text{i.e.} \quad \alpha = -\frac{1}{3}$$

by Proposition 2.1. (Or, we may use Viète’s formula, which is the relationship between roots and coefficients, to solve this problem.)

**Example 2.2.** Proposition 2.2 can be applied to the following problem:

‘Find roots of the quadratic equation  $2x^2 - 5x + 1 = 0$ .’

The solution without the quadratic formula is as follows:

On the hand, the discriminant of the given quadratic equation is

$$D = (-5)^2 - 4 \cdot 2 \cdot 1 = 17.$$

On the other hand, let  $\alpha$  be a root of the given quadratic equation. Then, the above discriminant can be also written as

$$D = (4\alpha - 5)^2$$

by Proposition 2.2. Hence, by solving

$$(4\alpha - 5)^2 = 17,$$

we have the roots

$$\alpha = \frac{5 \pm \sqrt{17}}{4}.$$

The quadratic formula is famous as one of mathematical formulas which are easy to forget. So, Proposition 2.2 is useful in the sense of that.

### 3. Cubic Equations, Functions and the Tangent Lines

We consider in this section a cubic equation

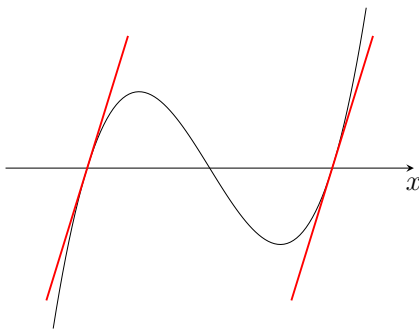
$$g(x) = ax^3 + bx^2 + cx + d = 0 \quad (3)$$

where  $a \in \mathbb{R} \setminus \{0\}$  and  $b, c, d \in \mathbb{R}$ . (There are many known ways to find the roots of (3), including the Tartaglia-Cardano formula.) It is well known that cubic functions have a lot of interesting properties. We here investigate whether the same proposition as Proposition 2.1 holds for (3). Recall the following terminology.

**Definition 3.1** ([2, 6, 7, 13, 15] etc.). Let  $F$  be a  $C^2$ -function. The points which are zero points of  $F''$  and are points where the sign of  $F''(x)$  changes before and after the points are called *inflection points* of  $F$ .

**Proposition 3.1.** Let  $g$  be a cubic function with all its zero points being single, and let  $\alpha$  (resp.  $\beta$ ) be the minimum (resp. maximum) zero point of  $g$ . We suppose that the inflection point of  $g$  is a zero point of  $g$ . Then, the slopes of two tangent lines at zero points  $\alpha, \beta$  of  $g$  are same signs, that is,

$$\frac{g'(\alpha)}{g'(\beta)} = 1.$$



**Figure 2.** Cubic function and the tangent lines.

*proof.* It is well known that any cubic function is symmetric with respect to the inflection point. Indeed, it is sufficient to see that  $\tilde{g}(x) = \tilde{a}x^3 + \tilde{c}x$  is symmetric with respect to the origin by translating of the inflection point to the origin. Hence, two tangent lines at zero points  $\alpha, \beta$  of  $g$  are parallel.

This completes the proof.

**Remark 3.1.** i) Proposition 3.1 is obviously nonsense or does not hold, if  $g$  has even one zero point with multiplicity of 2 or more.

ii) By virtue of Proposition 3.1, we can find that the tangent lines of  $g$  are parallel if they are symmetric with respect to the inflection point of  $g$ .

If the inflection point is not the zero point, Proposition 3.1 does not generally hold as we can see the following example.

**Example 3.1** (Cubic function whose inflection point is not the zero points). We see the counterexample of Proposition 3.1. Consider a cubic function

$$g(x) = (x - 10)(x - 1)(x + 1).$$

Then,  $g$  has three zero points  $-1, 1, 10$ . Since

$$g'(x) = (x - 1)(x + 1) + (x - 10)(x + 1) + (x - 10)(x - 1)$$

and

$$g''(x) = 2(3x - 10),$$

the inflection point of  $g$  is  $10/3$  and this is not the zero point of  $g$ . Moreover, we have

$$g'(-1) = 22 \neq 99 = g'(10).$$

### 4. $n$ th-degree Algebraic Equations, Functions and the Tangent Lines

Let us give general theorems which include Proposition 2.2 and Proposition 3.1. In order to do that, we mention the following easy fact in advance.

**Theorem 4.1.** Let  $n \geq 2$  and  $h$  be an  $n$ th-degree algebraic function. We suppose that  $\alpha$  (resp.  $\beta$ ) is the minimum (resp. maximum) zero point of  $h$ . Then,

1) if  $h$  is symmetric with respect to the  $y$ -axis, we have

$$\frac{h'(\alpha)}{h'(\beta)} = 1;$$

2) if  $h$  is symmetric with respect to the origin, we have

$$\frac{h'(\alpha)}{h'(\beta)} = -1.$$

*Proof.* The proof is trivial.

**Theorem 4.2.** Let  $n \geq 3$  be odd and  $h$  an  $n$ th-degree algebraic function. We suppose that  $\alpha$  (resp.  $\beta$ ) is the minimum (resp. maximum) zero point of  $h$ . If all the constants of even-power-terms of  $h$  are 0 (but the constant term does not have to be 0) and  $|\alpha| = |\beta|$ , then

$$\frac{h'(\alpha)}{h'(\beta)} = 1.$$

*proof.* From the assumption, we should consider

$$h(x) = \sum_{j=1}^{k+1} a_{2j-1} x^{2j-1}$$

where  $n = 2k + 1$  with  $k \geq 1$ . Then, since we have

$$h'(x) = \sum_{j=1}^{k+1} a_{2j-1} (2j-1) x^{2(j-1)},$$

it obviously holds that

$$h'(\alpha) = h'(\beta)$$

by virtue of another assumption ' $|\alpha| = |\beta|$ .' Hence, the proof has been finished.

*Remark 4.1.* i) As the proof shows, the important thing is  $h'$ . So, it is alright whether  $a_0 = 0$  or not because the constant term disappears by differentiating. We set  $a_0 = 0$  in the above proof.

ii) One of the assumptions ' $|\alpha| = |\beta|$ ' may be rewritten as ' $\alpha + \beta = 0$ .'

By the way, the following is known (See e.g. [1, 9, 11] and see [10] for more developed relationships between 4th-degree functions and their tangents). That is, suppose that the 4th-degree function  $h$  has two inflection points and satisfies the special condition for coefficients. Then,  $h - \ell$  has a vertical axis of symmetry where  $\ell$  is the straight line connecting two inflection points. Hence, we gain the following easy result for 4th degree.

*Proposition 4.1.* Let  $h$  be a 4th-degree function

$$h(x) = ax^4 + bx^3 + cx^2 + dx + e \quad (a \neq 0)$$

which has two inflection points P, Q. Suppose that  $3b^2 - 8ac > 0$ . If P and Q are zero points  $\alpha$  and  $\beta$  of  $h$  respectively, then

$$\frac{h'(\alpha)}{h'(\beta)} = -1.$$

*Example 4.1* (Function with inflection points that are not zero points, I). Consider the function

$$h_1(x) = (x-1)(x+1)(x-\sqrt{5})(x+\sqrt{5}).$$

Since

$$h'_1(x) = (x+1)(x-\sqrt{5})(x+\sqrt{5}) + (x-1)(x-\sqrt{5})(x+\sqrt{5}) \\ + (x-1)(x+1)(x+\sqrt{5}) + (x-1)(x+1)(x-\sqrt{5})$$

and

$$h''_1(x) = 12(x-1)(x+1),$$

$h_1$  has all inflection points as the zero points. Then, we have

$$h'_1(-\sqrt{5}) = -8\sqrt{5}, \quad h'_1(\sqrt{5}) = 8\sqrt{5}$$

for the maximum and minimum zero points  $\pm\sqrt{5}$  of  $h_1$ .

However,

$$h_2(x) = (x-1)x(x+1)^2$$

is not like that. Indeed,

$$h'_2(x) = x(x+2)^2 + (x-1)(x+1)^2 + 2(x-1)x(x+1), \\ h''_2(x) = 12x^2 + 10x + 1$$

imply that

$$|h'_2(-1)| = 1 \neq 9 = |h'_2(1)|$$

for the minimum and maximum zero points  $\pm 1$  of  $h_2$ .

*Example 4.2* (Function with inflection points that are not zero points, II). Consider the function

$$h(x) = x(x-1)(x^2 + x - 5).$$

Since

$$h'(x) = (x-1) \left( x - \frac{-1-\sqrt{21}}{2} \right) \left( x - \frac{-1+\sqrt{21}}{2} \right) \\ + x \left( x - \frac{-1-\sqrt{21}}{2} \right) \left( x - \frac{-1+\sqrt{21}}{2} \right) \\ + x(x-1) \left( x - \frac{-1+\sqrt{21}}{2} \right) \\ + x(x-1) \left( x - \frac{-1-\sqrt{21}}{2} \right)$$

and

$$h''(x) = 12(x-1)(x+1),$$

of the two inflection points, only 1 is the zero point. Hence, we have

$$|h'(\alpha)| \neq |h'(\beta)|$$

easily for minimum and maximum zero points  $(\alpha, \beta) = ((-1-\sqrt{21})/2, (-1+\sqrt{21})/2)$  of  $h$ .

*Example 4.3* ( $n$ th-degree algebraic function  $h$  which has  $n-2$  inflection points and not all inflection points are zero points). Consider the function

$$h(x) = (x-2)(x-1)(x+1)(x+3).$$

This  $h$  has two inflection points that is not zero points:

$$h'(x) = (x-1)(x+1)(x+3) + (x-2)(x+1)(x+3) \\ + (x-2)(x-1)(x+3) + (x-2)(x-1)(x+1)$$

$$h''(x) = 2(6x^2 + 3x - 7).$$

We can now obtain

$$|h'(-3)| = 40 \neq 15 = |h'(2)|,$$

since the maximum and minimum zero points of  $h$  are 2 and  $-3$  respectively.

Whereas, we can also obtain the following result for general

degrees.

**Theorem 4.3.** Let  $n \geq 3$  and  $h$  be an  $n$ th-degree algebraic function. We suppose that  $h$  has the real zero points

$$\alpha, \quad \alpha + \kappa, \quad \dots, \quad \alpha + (n-1)\kappa \quad (4)$$

for any  $\kappa \in \mathbb{R}$ . We write  $\beta$  for the maximum zero point of  $h$ :  $\beta = \alpha + (n-1)\kappa$ . The slopes of two tangent lines at zero points  $\alpha, \beta$  of  $h$  satisfy

$$\frac{h'(\alpha)}{h'(\beta)} = (-1)^{n-1}.$$

*proof.* Let

$$h(x) = \prod_{j=1}^n (x - \alpha_j)$$

be an  $n$ th-degree algebraic function where

$$\alpha_j = \alpha + (n-j)\kappa \quad (j = 1, \dots, n)$$

are real zero points of  $h$ . One has

$$h'(x) = \sum_{k=1}^n \frac{\prod_{j=1}^n (x - \alpha_j)}{x - \alpha_k}.$$

We obtain

$$h'(\alpha) = \prod_{j=2}^n (\alpha - \alpha_j) = \prod_{j=1}^{n-1} (-j\kappa) = (-1)^{n-1} \kappa^{n-1} (n-1)!$$

and

$$h'(\beta) = \prod_{j=1}^{n-1} (n-j)\kappa = \kappa^{n-1} (n-1)!$$

since

$$\beta - \alpha_l = (\alpha + (n-1)\kappa) - (\alpha + (l-1)\kappa) = (n-l)\kappa$$

for all  $l = 1, \dots, n$ . Hence, this completes the proof.

The assumed (4) is important for Theorem 4.3 to hold. This can be seen in the following example.

**Example 4.4** (Function which has non-equally spaced zero points). Consider two functions

$$\begin{aligned} h_1(x) &= (x-3)(x-1)(x+1)(x+3), \\ h_2(x) &= (x-3)(x-1)(x+1)(x+2). \end{aligned}$$

Then,  $h_1$  has four equally spaced zero points  $-3, -1, 1, 3$ . Since

$$\begin{aligned} h'_1(x) &= (x-1)(x+1)(x+3) + (x-3)(x+1)(x+3) \\ &\quad + (x-3)(x-1)(x+3) + (x-3)(x-1)(x+1), \end{aligned}$$

we have

$$h'_1(-3) = -48, \quad h'_1(3) = 48.$$

On the other hand,  $h_2$  has non-equally spaced zero points

$-2, -1, 1, 3$ , and since

$$\begin{aligned} h'_2(x) &= (x-1)(x+1)(x+2) + (x-3)(x+1)(x+2) \\ &\quad + (x-3)(x-1)(x+2) + (x-3)(x-1)(x+1), \end{aligned}$$

we have

$$|h'_2(-2)| = 15 \neq 40 = |h'_2(3)|.$$

Finally, we mention the relationship between inflection points being zero points and zero points being arranged at equal intervals. To conclude first, there is no reciprocity between these in general. Let us confirm that through counterexamples.

**Example 4.5.** Consider the function

$$h(x) = x^4 - 6x^2 - x + 5.$$

Since it is rewritten as

$$h(x) = (x-1)(x+1) \left( x - \frac{-1-\sqrt{21}}{2} \right) \left( x - \frac{-1+\sqrt{21}}{2} \right),$$

$h$  is the function whose zero points are not evenly spaced. On the other hand, inflection points of  $h$  are zero points of it, since

$$h''(x) = 12(x-1)(x+1).$$

Hence, even if the inflection points are zero points, it does not necessarily mean that the zero points are arranged at equal intervals.

**Example 4.6.** Consider the function

$$h(x) = (x-2)(x-1)x(x+1)$$

with zero points evenly spaced. Since

$$h''(x) = 2(6x^2 - 6x - 1),$$

none of the inflection points of  $h$  are not zero points. Hence, even if the zero points are arranged at equal intervals, it does not necessarily mean that the inflection points are zero points.

## 5. Conclusions

We have studied symmetry and asymmetry of tangent lines at maximum and minimum zero points of an  $n$ th-degree algebraic function ( $n \geq 2$ ). These properties are closely related to symmetry and asymmetry of the graph of the function. It is important to pay attention to symmetry of the object of discussion in mathematics. This applies not only to mathematics, but of course to science in general. We believe that our study in this note is part of it.

We tried to take up many examples (Examples 3.1, and 4.1-4.6) in order to grasp the core of our study in this note. Even (resp. Odd) number-degree functions cannot obviously have symmetry (resp. asymmetry) of the tangent lines at minimum and maximum zero points. On the other hand, it is not however easy to see if even (resp. odd) number-degree

functions have asymmetry (resp. symmetry) of them. One of answers (sufficient, or, necessary and sufficient conditions) for this problem is Theorem 4.3. (Theorem 4.2 is also one answer to the case of odd number-degree.) The existence of other answers is also, of course, expected. We would like to make that a future topic.

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