

On Discrete Functions and Repetitive Arrangements with Algorithms to Construct All Discrete Functions and a Practical Problem

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Abstract: The main idea of this work is based on the question: how can we control the electric circuits between a number of electric bulbs and a number of electric sources. This generates the correspondences between two discrete sets. The correspondence is based on the notion of discrete function and repetitive arrangements. The normal construction and notions of the work are introduced gradually and are detailed at every stage. Our constant endeavour has been to ensure that every sentence in the work has a logical position. Here appears many questions: how to construct all discrete functions, which is the total number of these functions, which is the relation between the number of bulbs and the number of sources, can we construct and control only a partial number of electric circuits (by direct access method) etc. The work answers all these questions by specialised algorithms: the construction algorithm and the decomposition algorithm. The algorithms use the rule from left to right to construct all possible discrete functions and, hence, all electric circuits. The decomposition algorithm supplies an access direct method. So we can control any part of the whole set of circuits. A lot of notions and specific notations are used to develop and illustrate the work. For combinations we have to show the construction elements. A lot of examples explain this important notion. The work contains a lot of numerical examples and applications. The last section of the work deals with the bijective (and invertible) functions. Specialized notions and notations are used. Numerical examples and geometric designs illustrate the theory.

Keywords: Arrangements, Repetitive Arrangements, Discrete Functions, Decomposition Algorithm, Rule Left Right

1. Introduction

How to control k electric light bulbs $\{a_1, a_2, \dots, a_k\}$ alimented (supplied) by n $\{b_1, b_2, \dots, b_n\}$ sources of electricity. To control it means to know the form of electric circuit, namely the correspondence between these two sets.

In order to solve the above problem we use the discrete functions [1, 3, 8] and we use some specific notations and remarks.

We denote $f: A \rightarrow B$, A -domain, B -codomain; with the discrete sets

$$A = \{a_1, a_2, \dots, a_k\}, B = \{b_1, b_2, \dots, b_n\}$$

for any natural non-null numbers k and n ; $\text{card } A = |A| = k$, $\text{card } B = |B| = n$, $a_i \neq a_j$ for any i and j . The set B is a multiple set [10].

The set of all functions $f: A \rightarrow B$ has the cardinal

$$\text{card } \{f\} = \text{card } B^{\text{card } A} = n^k \quad (1)$$

In applications we use $A = \{1, 2, \dots, i, \dots, k\}$, $B = \{1, 2, \dots, j, \dots, n\}$.

There are three cases: $k < n$, $k = n$, $k > n$ [10, 13].

The injective functions f could be obtained for $k \leq n$.

The surjective functions f could be obtained for $k \geq n$.

The bijective functions f could be obtained for $k = n$.

Our aim is to construct all n^k functions $f: A \rightarrow B$. Then we will analyse the bijective functions [5].

Remark 1. Permutations and arrangements are ordinate sets, while combinations are subsets of a set.

We present a short comparison between the usual arrangements of n objects taken k at a time and the repetitive arrangements (arrangements with repetition) of n objects taken k at a time [6, 7].

Usual arrangements. Example 1. $B = \{1, 2, 3\}$; $n = 3$, $k = 2$; $n \geq k$ (always).

Permutations $n = 3$; 123; 132; 213; 231; 312; 321; $n! = 3! = 6$.

Arrangements of $n = 3$ taken $k = 2$; 12; 21; 13; 31; 23; 32; $A_n^k = A_3^2 = 6$.

Combinations $n = 3$ taken $k = 2$; 12; 13; 23; $C_n^k = C_3^2 = 3$.

Repetitive arrangements [2, 11, 12]. Example 2. $B = \{1, 2, 3\}$; $n = 3$, $k = 2$ (any value n, k)

11; 12; 13; 21 22; 23;; 31; 32; 33; $n^k = 3^2 = 9$. Denote $n^k = N$.

Example 3. $B = \{1, 2\}$; $n = 2$, $k = 3$; $n \leq k$.

111; 112; 121; 211; 122; 221; 212; 222; $n^k = 2^3 = 8$.

Another notation for the total number of all repetitive

arrangements is $\overline{A}_n^k = n^k = a_n^k = N$.

2. Problem Formulation

Related with this work we have two aims.

1) Aim 1. We have to construct the set of all discrete functions $f: A \rightarrow B$,

$f: \{a_1, a_2, \dots, a_k\} \rightarrow \{b_1, b_2, \dots, b_n\}$ for any non-zero natural numbers n, k .

The total number of these functions is denoted $\overline{A}_n^k = n^k = N$.

There are several methods to construct all discrete functions f .

We propose a method based on direction left to right in the set B , and elaborate the algorithm left – right (algorithm 1). It is a sequential method.

2) Aim 2. We make a decomposition of $\overline{A}_n^k = n^k$ based on the decomposition algorithm (algorithm 2). It is a direct acces method, i.e. we can construct any subset of the hole set with $\overline{A}_n^k = n^k$ functions.

Examples. Some particular cases and the total number of functions.

| Nr | k | n | n^k | $f: A \rightarrow B$ | |
|------|-----|-----|-------|---|--------------|
| 1 | 1 | 2 | nu | $1 \rightarrow 1, 2$ | $N = 2$. |
| 2 | 2 | 1 | 1 | $1, 2 \rightarrow 1$ | $N = 1$. |
| 3 | 2 | 2 | 4 | $1, 2 \rightarrow 1, 2$ | $N = 4$. |
| 4 | 3 | 2 | 8 | $1, 2, 3 \rightarrow 1, 2$ | $N = 8$. |
| 5 | 4 | 2 | 16 | $1, 2, 3, 4 \rightarrow 1, 2$ | $N = 16$. |
| 6 | 5 | 2 | 32 | $1, 2, 3, 4, 5 \rightarrow 1, 2$ | $N = 32$. |
| 7 | 2 | 3 | 9 | $1, 2 \rightarrow 1, 2, 3$ | $N = 9$. |
| 8 | 3 | 3 | 27 | $1, 2, 3 \rightarrow 1, 2, 3$ | $N = 27$. |
| 9 | 4 | 3 | 81 | $1, 2, 3, 4 \rightarrow 1, 2, 3$ | $N = 81$. |
| 10 | 5 | 3 | 243 | $1, 2, 3, 4, 5 \rightarrow 1, 2, 3$ | $N = 243$. |
| 11 | 2 | 4 | 16 | $1, 2 \rightarrow 1, 2, 3, 4$ | $N = 16$. |
| 12 | 3 | 4 | 64 | $1, 2, 3 \rightarrow 1, 2, 3, 4$ | $N = 48$. |
| 13 | 4 | 4 | 256 | $1, 2, 3, 4 \rightarrow 1, 2, 3, 4$ | $N = 256$. |
| 14 | 5 | 4 | 1024 | $1, 2, 3, 4, 5 \rightarrow 1, 2, 3, 4$ etc. | $N = 1024$. |

Remark 2. Tthe total number of functions increases very quicly with k and n .

3. The Algorithm Left - Right to Construct All Discrete Functions

The functions are $f: A \rightarrow B$, $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_n\}$, where $k \geq 1$ and $n \geq 1$ are given natural numbers.

The total numnner of functions f is the total number of repetitive arrangements.

The rule from left to right [7] means: if the fix value is j or b_j , then, for combinations one uses only the values $S = \{b_{j+1}, b_{j+2}, \dots, b_n\}$ from the right part of b_j .

Remark 3. The algorithm is based on combinations and the moving rule down-up.

Example.

1) $1, 2, 3 \rightarrow 1, 2, 3$; $C_3^1 = 3$; $f(1) = 2 \ 1 \ 3$;
 $f(2) = 1 \ 3 \ 2$;
 $f(3) = 3 \ 2 \ 1$.

2) $1, 2, 3 \rightarrow a, b, c, d$; $C_3^1 = 3$; $f(1) = b \ b \ d$;
 $f(2) = b \ d \ b$;
 $f(3) = d \ b \ b$.

3.1. Examples of All Discrete Functions for Small Numbers k and n

Application 3.1. We use the algorithm 1. It is a sequential method. (Version 1).

$$1) f: 1, 2 \rightarrow 1, 2, N = 2^2 = 4$$

| | | | | |
|----------|---|---|---|---|
| Nr | 1 | 2 | 3 | 4 |
| $f(1) =$ | 1 | 1 | 2 | 2 |
| $f(2) =$ | 1 | 2 | 1 | 2 |

$$2) f: 1, 2, 3 \rightarrow 1, 2, N = 2^3 = 8$$

| | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $f(1) =$ | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 |
| $f(2) =$ | 1 | 1 | 2 | 1 | 2 | 2 | 1 | 2 |
| $f(3) =$ | 1 | 2 | 1 | 1 | 2 | 1 | 2 | 2 |

On the line Nr we count the current number of function f or its address.

$$3) f: 1, 2, 3, 4 \rightarrow 1, 2, N = 2^4 = 16 \text{ functions (small number).}$$

| | | | | | | | | | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|----|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 16 |
| $f(1) =$ | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 2 |
| $f(2) =$ | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 2 | 2 | 1 | 2 |
| $f(3) =$ | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 2 | 2 |
| $f(4) =$ | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 2 | 2 |

$$4) f: 1, 2 \rightarrow 1, 2, 3, 4, N = 4^2 = 16 \text{ functions (small number).}$$

| | | | | | | | | | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|----|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 16 |
| $f(1) =$ | 1 | 1 | 2 | 1 | 3 | 1 | 4 | 2 | 2 | 3 | 2 | 4 | 3 | 3 | 4 | 4 |
| $f(2) =$ | 1 | 2 | 1 | 3 | 1 | 4 | 1 | 2 | 3 | 2 | 4 | 2 | 3 | 4 | 3 | 4 |

$$5) f: 1, 2, 3, 4, 5 \rightarrow 1, 2, N = 2^5 = 32 \text{ functions (small number).}$$

| | | | | | | | | | | | | | | | | | | | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|---|---|---|---|---|----|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 16 | 17 | 18 | 19 | 0 | 1 | 2 | 3 | 4 | 5 | 26 |
| $f(1) =$ | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 |
| $f(2) =$ | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 |
| $f(3) =$ | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 1 |
| $f(4) =$ | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 |
| $f(5) =$ | 1 | 2 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 2 |

| | | | | | | |
|----------|----|---|---|---|---|----|
| Nr | 27 | 8 | 9 | 0 | 1 | 32 |
| $f(1) =$ | 1 | 2 | 2 | 2 | 2 | 2 |
| $f(2) =$ | 2 | 2 | 2 | 2 | 1 | 2 |
| $f(3) =$ | 2 | 2 | 2 | 1 | 2 | 2 |
| $f(4) =$ | 2 | 2 | 1 | 2 | 2 | 2 |
| $f(5) =$ | 2 | 1 | 2 | 2 | 2 | 2 |

$$6) f: 1, 2, 3 \rightarrow 1, 2, 3, N = 3^3 = 27 \text{ functions (small number).}$$

| | | | | | | | | | | | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|----|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 27 |
| $f(1) =$ | 1 | 1 | 1 | 2 | 1 | 3 | 1 | 2 | 3 | 1 | 3 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| $f(2) =$ | 1 | 1 | 2 | 1 | 3 | 1 | 2 | 2 | 3 | 1 | 3 | 2 | 1 | 2 | 3 | 3 | 2 | 3 |
| $f(3) =$ | 1 | 2 | 1 | 3 | 1 | 1 | 2 | 1 | 2 | 3 | 1 | 3 | 2 | 1 | 2 | 3 | 2 | 3 |

Remark 4. For $k=3$ and $n=3$ a short summary of the basic idea of computations has the following form

| | | | | | | |
|----------|-----|-----|-----|-----|-----|-----|
| $f(1)=1$ | : 1 | : 1 | : 2 | : 2 | : 2 | : 3 |
| $f(2)=1$ | : 1 | : x | : 2 | : 2 | : x | : 3 |
| $f(3)=1$ | : x | : y | : 2 | : x | : y | : 3 |

$$C_3^3(1; \theta) ; C_3^2(1; x) ; C_3^1(1; x, y) ; C_3^3(2; \theta) ; C_3^2(2; x) ; C_3^1(2; x, y) ; C_3^3(3; \theta)$$

The symbol θ is empty set and $x, y, \dots \in S$.

3.2. Examples of All Discrete Functions for Great Numbers k and n

We use a modified sequential method. (Version 2) [4, 5].

Use successively the rules a), b), c), d), e).

Use the subset $S \subset B$ with all elements from the right part of j .

Denote by L the completion length, $0 \leq L \leq k-1$.

Construct all repetitive arrangements of S having the length L , $L = 0, 1, 2, \dots$ etc.

Compute the total number of arrangements $(n-j)^L$.

Denote $(n-j)^L = R$.

Increase the last address with the value $R \times C_k^{k-L}$.

Construct all bijective functions corresponding to the number $(n-j)^L$.

Application 3.2.1. $f: 1, 2, 3 \rightarrow 1, 2, 3, 4, 5, k=3; n=5$;

$N = 5^3 = 125$ functions (great number).

Step $j=1, L=0; C_k^{k-L} = C_3^{3-0} = 1$.

Nr 1 (first address)

$$f(1)=1$$

$$f(2)=1$$

$$f(3)=1$$

$$j=1, L=1, S=\{2, 3, 4, 5\}, (n-j)^L = (5-1)^1 = 4 = R ;$$

use 2 3 4 5

and the arrangements 2, 3, 4, 5; $C_k^{k-L} = C_3^{3-1} = 3$.

Increase the last address 1 with the value $R \times 3 = 12$; $1+12=13$; true.

Nr 2 3 4 5 6 7 8 9 10 11 12 13 (addresses)

$$f(1)=1 1 2 1 1 1 1 1 1 5$$

$$f(2)=1 2 1 1 1 1 1 5 1 1$$

$$f(3)=2 1 2 3 4 5 1 1$$

$$j=1, L=2, S=\{2, 3, 4, 5\}, (n-j)^L = (5-1)^2 = 16 = R ; \text{ use } 2 3 4 5$$

and the arrangements

$$2 2 2 2 3 3 3 3 4 4 4 4 5 5 5 5 ; C_k^{k-L} = C_3^{3-2} = 3.$$

$$2 3 4 5 2 3 4 5 2 3 4 5 2 3 4 5 ;$$

Increase the last address 13 with the value $R \times 3 = 48$; $13+48=61$; true.

Nr 14 15 16 17 18 19 20 21 22 23 24 25 56 57 58 59 60 61 (addresses)

$$f(1)=1 2 2 1 1 1 1 1 1 3 2 1 1 5 4 1 5 5 5$$

$$f(2)=2 2 1 2 2 2 3 2 1 5 4 1 5 5 1$$

$$f(3)=2 1 2 3 4 4 2 1 3 4 1 5 5 1 5$$

Step $j=2$, $L=0$. $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 62 (new address)

$f(1)=2$

$f(2)=2$

$f(3)=2$

$j=2$, $L=1$, $S=\{3, 4, 5\}$, $(n-j)^L = (5-2)^1 = 3 = R$; use 3, 4, 5

and the arrangements 3, 4, 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 62 with the value $R \times 3=9$; $62+9=71$; true.

Nr 63 64 65 66 67 68 69 70 71 (addresses)

$f(1)=2$ 2 3 2 2 2 5

$f(2)=2$ 3 2 2 2 5 2

$f(3)=3$ 2 2 4 5 2 2

$j=2$, $L=2$, $S=\{3, 4, 5\}$, $(n-j)^L = (5-2)^2 = 9 = R$, use 3, 4, 5 and the arrangements

3 3 3 4 4 4 5 5 5; $C_k^{k-L} = C_3^{3-L} = 3$.

3 4 5 3 4 5 3 4 5;

Increase the last address 71 with the value $R \times 3=9$; $71+9=80$; true.

Nr 72 73 74 75 76 77 78 79 80 93 94 95 96 97 98 (addresses)

$f(1)=2$ 3 3 2 2 3 5 2 2 5 5

$f(2)=3$ 3 2 3 3 5 2 5 5 5 2

$f(3)=3$ 2 3 4 5 2 3 4 5 2 5

Step $j=3$, $L=0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 99 (new address)

$f(1)=3$

$f(2)=3$

$f(3)=3$

$j=3$, $L=1$, $S=\{4, 5\}$, $(n-j)^L = (5-3)^1 = 2 = R$; use 4, 5

and the arrangements 4, 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 99 with the value $R \times 3=6$; $99+6=105$; true.

Nr 100 101 102 103 104 105 (addresses)

$f(1)=3$ 3 4 3 3 5

$f(2)=3$ 4 3 3 5 3

$f(3)=4$ 3 3 5 3 3

$j=3$, $L=2$, $S=\{4, 5\}$, $(n-j)^L = (5-3)^2 = 4 = R$; use 4, 5

and the arrangements 4 4 5 5; $C_k^{k-L} = C_3^{3-L} = 3$.

4 5 4 5;

Increase the last address 105 with the value $R \times 3=12$; $105+12=117$; true.

Nr 106 107 108 109 110 111 112 113 114 115 116 117 (addresses)

$f(1)=3$ 4 4 3 4 5 3 3 5 5

$f(2)=4$ 4 3 4 5 3 5 5 5 3

$f(3)=4$ 3 4 5 3 4 4 5 3 5

Step $j=4$, $L=0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 118 (new address)

$f(1)=4$

$f(2)=4$

$f(3)=4$

$j=4$, $L=1$, $S=\{5\}$, $(n-j)^L = (5-4)^1 = 1 = R$; use 5

and the arrangements 5; $C_k^{k-L} = C_3^{3-L} = 3$.

Increase the last address 118 with the value $R \times 3=3$; $118+3=121$; true.

Nr 119 120 121 (addresses)

$f(1)=4$ 4 5

$f(2)=4$ 5 4

$f(3)=5$ 4 4

$j=4$, $L=2$, $S=\{5\}$, $(n-j)^L = (5-4)^2 = 1 = R$; use 5

and the arrangements 5; $C_k^{k-L} = C_3^{3-L} = 3$.

5;

Increase the last address 121 with the value $R \times 3=3$; $121+3=124$; true.

Nr 122 123 124 (addresses)

$f(1)=4$ 5 5

$f(2)=5$ 5 4

$f(3)=5$ 4 5

Step $j=$, $L=0$; $C_k^{k-L} = C_3^{3-L} = 1$.

Nr 125 (last address). End of computing. Stop. $N=125$.

$f(1)=5$

$f(2)=5$

$f(3)=5$

Application 3.2.2. $f: 1, 2, 3, 4, 5 \rightarrow 1, 2$, $N=32$

Nr 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 16 17 8 9 0 1 2 3 4 5 6=26

$f(1)=$ 1 1 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 2 1 2 2

$f(2)=$ 1 1 1 1 2 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 2 1 2 2 1

$f(3)=$ 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 1 1 2 2 1 2

$f(4)=$ 1 1 2 1 1 1 2 2 1 1 1 1 2 1 1 2 2 2 1 1 2 2 2 1 2 1

$f(5)=$ 1 2 1 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 2 2 2 1 2 1 2

Nr 27 8 9 0 1 2=32

$f(1)=$ 1 2 2 2 2

$f(2)=$ 2 2 2 2 1

$f(3)=$ 2 2 2 1 2

$f(4)=$ 2 2 1 2 2

$f(5)=$ 2 1 2 2 2

$$N = C_5^5(1; \theta) + C_5^4(1; 2) + C_5^3(1; 2, 2) + C_5^2(1; 2, 2, 2) + C_5^1(1; 2, 2, 2, 2) + C_5^5(2; \theta)$$

$N=1+5+10+10+5+1=32$ positions=32 functions.

| | | | | | | | | | | | | | | | | | |
|---------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|------|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7=27 |
| $f(1)=$ | 1 | 1 | 1 | 2 | 1 | 1 | 3 | 1 | 2 | 2 | 1 | 2 | 3 | 1 | 3 | 2 | 3 |
| $f(2)=$ | 1 | 1 | 2 | 1 | 1 | 3 | 1 | 2 | 2 | 1 | 2 | 3 | 1 | 3 | 2 | 3 | 3 |
| $f(3)=$ | 1 | 2 | 1 | 1 | 3 | 1 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 2 | 1 | 3 | 3 |

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|--|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| $f(1)=$ | 1 | 1 | 1 | 2 | 1 | 1 | 3 | 1 | 1 | 4 | 1 | 2 | 2 | 1 | 2 | 3 | 1 | 2 | 4 | 1 | 3 | 2 | 1 | 3 | 3 | 1 | 3 | 4 | 1 | 4 | 2 | 1 | 4 | 3 | 1 | 4 | 4 | | |
| $f(2)=$ | 1 | 1 | 2 | 1 | 1 | 3 | 1 | 1 | 4 | 1 | 2 | 2 | 1 | 2 | 3 | 1 | 2 | 4 | 1 | 3 | 2 | 1 | 3 | 3 | 1 | 3 | 4 | 1 | 4 | 2 | 1 | 4 | 3 | 1 | 4 | 4 | 1 | | |
| $f(3)=$ | 1 | 2 | 1 | 1 | 3 | 1 | 1 | 4 | 1 | 1 | 2 | 1 | 2 | 3 | 1 | 2 | 4 | 1 | 2 | 2 | 1 | 3 | 3 | 1 | 3 | 4 | 1 | 3 | 2 | 1 | 4 | 3 | 1 | 4 | 4 | 1 | 4 | 1 | |

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|-----|---|---|---|---|---|---|---|---|---|------|
| Nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | =37 | | | | | | | | | | |
| $f(1)=1$ | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 2 | 3 | 1 | 2 | 1 | 1 | 3 | 2 | 1 | 3 | 1 | 1 | 2 | 2 | 2 | | | | | | |
| $f(2)=1$ | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 2 | 3 | 1 | 2 | 1 | 1 | 3 | 2 | 1 | 3 | 1 | 2 | 1 | 3 | 3 | 1 | 3 | 1 | 2 | 2 | 2 | 1 |
| $f(3)=1$ | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | 2 | 3 | 1 | 1 | 1 | 3 | 3 | 2 | 1 | 1 | 1 | 2 | 1 | 3 | 3 | 1 | 1 | 1 | 3 | 2 | 2 | 1 | 2 | |
| $f(4)=1$ | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 3 | 1 | 1 | 2 | 3 | 1 | 2 | 1 | 1 | 3 | 2 | 1 | 3 | 3 | 1 | 1 | 3 | 3 | 1 | 2 | 1 | 2 | 1 | etc. |

$$N = 1 + [8] + [24] + [32] + 1 + 4 + 6 + 4 + 1 = 81 \text{ positions} = 81 \text{ functions. True}$$

4. General Case for Left – Right Algorithm and Decomposition Algorithm

The bijective functions are $f: A \rightarrow B$,
 $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_n\}$.

All combinations have the form C_k^i , $i = 1, k$; C_k^k , C_k^{k-1} ,
 $C_k^{k-2}, \dots, C_k^1 = k$; $C_k^0 = 0$.

and the function f is defined for $f(a_1)$, $f(a_2)$, ..., $f(a_k)$.

Remark 5. [7, 9]

$C_k^k(b_1; \theta)$, $k + \text{card}\{\theta\} = k$;

$C_k^{k-1}(b_1; x)$, for $x: b_2, b_3, \dots, b_n$; $k - 1 + \text{card}\{x\} = k$;

$C_k^{k-2}(b_1; x, y)$; for $x, y: b_2, b_2; b_2, b_3; \text{etc.}$; $k - 2 + \text{card}\{x, y\} = k$;

$C_k^{k-3}(b_1; x, y, z)$, $k - 3 + \text{card}\{x, y, z\} = k$ etc.

We introduce some adequate notations and we analyze the algorithm for $k = 4$ and $n = 4$, $N = 256$. (Version 3).

Number m is used for combinations C_k^m ,
 $m = k, k-1, \dots, 3, 2, 1$.

j or b_j is the current number from $C_k^m(b_j; \dots)$, where
 $m \leq k$.

q indicates the number of completion elements taken from
the set B in arrangements, with $m + q = k$.

p indicates the total elements from set
 $B = \{b_1, b_2, \dots, b_j, b_{j+1}, \dots, b_n\}$ having the position in the
right of b_j ; $p = n - j$. We include the parameters p and q
in the above notation and obtain

$$C_k^m(b_j; p; q). \quad (2)$$

Verification:

$$p = n - j, \quad m + q = k \quad (3)$$

p^q is the partial number of repetitive arrangements for

$$m, b_j, p, q \quad (4)$$

$T(b_j; p; q)$ (natural number) is the last partial address
generated by combinations

$$C_k^m(b_j; p; q).$$

$$T(b_j; p; q) = C_k^m p^q \quad (5)$$

The sum of all $T(b_j; p; q)$ is N . [13, 14]

Application 4.1. Input data: $k = 4$ and $n = 4$, $N = 256$;
 $f: 1, 2, 3, 4 \rightarrow 1, 2, 3, 4$.

For verification we use the formulas (1), (2), (3), (4).

$$C_4^4(1; \theta) = 1, \quad T(1; \theta) = 1.$$

$$C_4^3(1; p = 3; q = 1) = 4, \quad p^q = 3, \quad T(1; p = 3; q = 1) = C_4^3 p^q = 4 \times 3 = 12 \text{ (product)}.$$

$p = 3, q = 1 \Rightarrow$ completion values are 2, 3, 4.

$$C_4^2(1; p = 3; q = 2) = 6, \quad p^q = 9, \quad T(1; p = 3; q = 2) = C_4^2 p^q = 6 \times 9 = 54.$$

$p = 3, q = 2 \Rightarrow$ completion values are 22, 23, 24; 32, 33, 34; 42, 43, 44.

$$C_4^1(1; p = 3; q = 3) = 4, \quad p^q = 27, \quad T(1; p = 3; q = 3) = C_4^1 p^q = 4 \times 27 = 108.$$

$p = 3, q = 3 \Rightarrow$ completion values are 222, 223, 224; 232, 233, 234; 242, 243, 244 etc.

$$C_4^4(2; \theta) = 1, \quad T(2; \theta) = 1.$$

$$C_4^3(2; p = 2; q = 1) = 4, \quad p^q = 2, \quad T(2; p = 2; q = 1) = C_4^3 p^q = 4 \times 2 = 8.$$

$p = 2, q = 1 \Rightarrow$ completion values are 3, 4.

$$C_4^2(2; p = 2; q = 2) = 6, \quad p^q = 4, \quad T(2; p = 2; q = 2) = C_4^2 p^q = 6 \times 4 = 24.$$

$p = 2, q = 2 \Rightarrow$ completion values are, 33, 34; 43, 44.

$$C_4^1(2; p = 2; q = 3) = 4, \quad p^q = 8, \quad T(2; p = 2; q = 3) = C_4^1 p^q = 4 \times 8 = 32.$$

$p = 2, q = 3 \Rightarrow$ completion values are 333, 334; 343, 344; 433, 434; 443, 444.

$$C_4^4(3; \theta) = 1, T(3; \theta) = 1.$$

$$C_4^3(3; p=1; q=1) = 4, p^q = 1, T(3; p=1; q=1) = C_4^3 p^q = 4 \times 1 = 4.$$

$p=1, q=1 \Rightarrow$ completion values are 4.

$$C_4^2(3; p=1; q=2) = 6, p^q = 1, T(3; p=1; q=2) = C_4^2 p^q = 6 \times 1 = 6.$$

$$C_4^1(3; p=1; q=3) = 4, p^q = 1, T(3; p=1; q=3) = C_4^1 p^q = 4 \times 1 = 4.$$

$p=1, q=3 \Rightarrow$ completion values are 444.

$$C_4^4(4; \theta) = 1, T(4; \theta) = 1.$$

Verification.

$$\begin{aligned} &T(1; \theta) + T(1; p=3; q=1) + T(1; p=3; q=2) + T(1; p=3; q=3) + T(2; \theta) + T(2; p=2; q=1) \\ &+ T(2; p=2; q=2) + T(2; p=2; q=3) + T(3; \theta) + T(3; p=1; q=1) + T(3; p=1; q=2) + T(3; p=1; q=3) + T(4; \theta) = N. \\ &+ T(3; p=1; q=2) + T(3; p=1; q=3) + T(4; \theta) = N \end{aligned}$$

$1+12+54+108+1+8+24+32+1+4+6+4+1=256; N=256$. True.

Application 4.2. $f: 1, 2, 3, 4 \rightarrow 1, 2, 3, 4, N=256$

We describe only the decomposition of repetitive arrangements.

There are a number of 60 groups of combinations.

$$\begin{aligned} N = &C_4^4(1; \theta) + [C_4^3(1; 2) + C_4^3(1; 3) + C_4^3(1; 4)] + [C_4^2(1; 2, 2) + C_4^2(1; 2, 3) + C_4^2(1; 2, 4) + \\ &+ C_4^2(1; 3, 2) + C_4^2(1; 3, 3) + C_4^2(1; 3, 4) + C_4^2(1; 4, 2) + C_4^2(1; 4, 3) + C_4^2(1; 4, 4)] + \\ &+ [C_4^1(1; 2, 2, 2) + C_4^1(1; 2, 2, 3) + C_4^1(1; 2, 2, 4) + C_4^1(1; 2, 3, 2) + C_4^1(1; 2, 3, 3) + \\ &+ C_4^1(1; 2, 3, 4) + C_4^1(1; 2, 4, 2) + C_4^1(1; 2, 4, 3) + C_4^1(1; 2, 4, 4) + C_4^1(1; 3, 2, 2) + \\ &+ C_4^1(1; 3, 2, 3) + C_4^1(1; 3, 2, 4) + C_4^1(1; 3, 3, 2) + C_4^1(1; 3, 3, 3) + C_4^1(1; 3, 3, 4) + \\ &+ C_4^1(1; 3, 4, 2) + C_4^1(1; 3, 4, 3) + C_4^1(1; 3, 4, 4) + C_4^1(1; 4, 2, 2) + C_4^1(1; 4, 2, 3) + \\ &+ C_4^1(1; 4, 2, 4) + C_4^1(1; 4, 3, 2) + C_4^1(1; 4, 3, 3) + C_4^1(1; 4, 3, 4) + C_4^1(1; 4, 4, 2) + \\ &+ C_4^1(1; 4, 4, 3) + C_4^1(1; 4, 4, 4)] + C_4^4(2; \theta) + [C_4^3(2; 3) + C_4^3(2; 4)] + \\ &+ [C_4^2(2; 3, 3) + C_4^2(2; 3, 4) + C_4^2(2; 4, 3) + C_4^2(2; 4, 4)] + \\ &+ [C_4^1(2; 3, 3, 3) + C_4^1(2; 3, 3, 4) + C_4^1(2; 3, 4, 3) + C_4^1(2; 3, 3, 4) + C_4^1(2; 4, 3, 3) + \\ &+ C_4^1(2; 4, 3, 4) + C_4^1(2; 4, 4, 3) + C_4^1(2; 4, 4, 4)] + C_4^4(3; \theta) + C_4^3(3; 4) + \\ &+ C_4^2(3; 4, 4) + C_4^1(3; 4, 4, 4) + C_4^4(4; \theta). \end{aligned}$$

$N=1+(4 \times 3)+(6 \times 9)+(4 \times 27)+1+(4 \times 2)+(6 \times 4)+(4 \times 8)+1+4+6+4+1=256$ positions, 256 functions.

bulbs and 2 sources. The model is $f: \{a_1, a_2, a_3\} \rightarrow \{b_1, b_2\}$ or $f: \{1, 2, 3\} \rightarrow \{1, 2\}; N=8$.

5. The Discrete Functions Applied to the Practical Problem

Application 5.1 Construct all electric circuits for 3 electric

Solution. Nr 1 2 3 4 5 6 7 8 (circuit addresses)
 $f(1) = 1 \ 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ 2$
 $f(2) = 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2$
 $f(3) = 1 \ 2 \ 1 \ 1 \ 2 \ 1 \ 2 \ 2$

The figure 1 shows all the circuits.

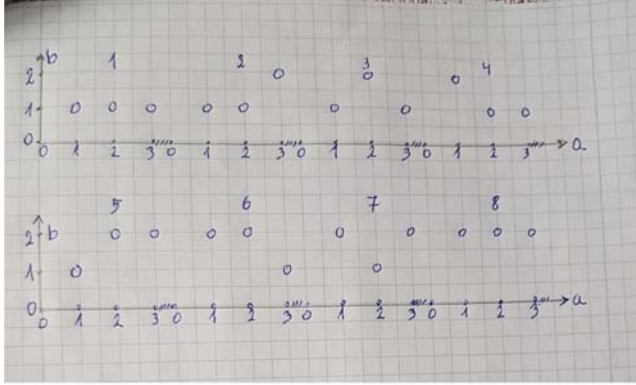


Figure 1. Electric circuits from application 5.1.

Application 5.2 Construct all electric circuits for 2 electric bulbs and 3 sources. The model is $f: \{a_1, a_2\} \rightarrow \{b_1, b_2, b_3\}$ or $f: \{1, 2\} \rightarrow \{1, 2, 3\}$; $N = 9$.

Solution. Nr 1 2 3 4 5 6 7 8 9 (circuit addresses)
 $f(1) = 1 \ 1 \ 2 \ 1 \ 3 \ 2 \ 2 \ 3 \ 3$
 $f(2) = 1 \ 2 \ 1 \ 3 \ 1 \ 2 \ 3 \ 2 \ 3$

The figure 2 shows all the circuits from application 5.2, by using the orthogonal axis O a b.

All other cases generates electrical circuits like the above circuits.

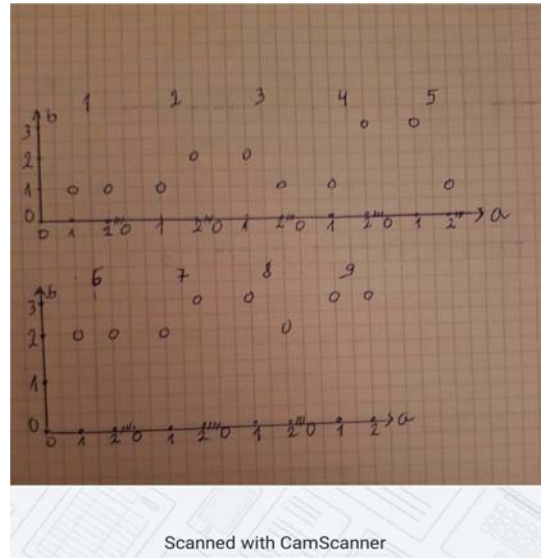


Figure 2. Electric circuits from application 5.2.

6. Bijective Functions

We use $k = n$; $f: \{a_1, a_2, \dots, a_n\} \rightarrow \{b_1, b_2, \dots, b_n\}$ $n \geq 1$.

The total number of all bijective functions is $N_1 = n!$.

Application 6.1. $f: 1, 2, 3 \rightarrow 1, 2, 3$; $N_1 = 3! = 6$;

$$C_3^1(1, 2, 3) + C_3^1(1, 3, 2) = 3 + 3 = 6.$$

Nr 1 2 3 4 5 6 (addresses)
 $f(1) = 1 \ 2 \ 3 \ 1 \ 3 \ 2$
 $f(2) = 2 \ 3 \ 1 \ 3 \ 2 \ 1$
 $f(3) = 3 \ 1 \ 2 \ 2 \ 1 \ 3$.

Application 6.2. $f: 1, 2, 3, 4 \rightarrow 1, 2, 3, 4$; $N_1 = 4! = 24$ (bijective functions).

$$\begin{aligned} &C_4^1(1, 2, 3, 4) + C_4^1(1, 2, 4, 3) + C_4^1(1, 3, 2, 4) + C_4^1(1, 3, 4, 2) + \\ &C_4^1(1, 4, 2, 3) + C_4^1(1, 4, 3, 2) = \\ &+ 4 + 4 + 4 + 4 + 4 + 4 = 24; C_4^1 = 4. \end{aligned}$$

Nr 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $f(1) = 1 \ 2 \ 3 \ 4 \ 1 \ 2 \ 4 \ 3 \ 1 \ 3 \ 2 \ 4 \ 1 \ 3 \ 4 \ 2 \ 1 \ 4 \ 2 \ 3 \ 1 \ 4 \ 3 \ 2 \ 1$
 $f(2) = 2 \ 3 \ 4 \ 1 \ 2 \ 4 \ 3 \ 1 \ 3 \ 2 \ 4 \ 1 \ 3 \ 4 \ 2 \ 1 \ 4 \ 2 \ 3 \ 1 \ 4 \ 3 \ 2 \ 1$
 $f(3) = 3 \ 4 \ 1 \ 2 \ 4 \ 3 \ 1 \ 2 \ 2 \ 4 \ 1 \ 3 \ 4 \ 2 \ 1 \ 3 \ 2 \ 3 \ 1 \ 3 \ 3 \ 2 \ 1 \ 4$
 $f(4) = 4 \ 1 \ 2 \ 3 \ 3 \ 1 \ 2 \ 4 \ 4 \ 1 \ 3 \ 2 \ 2 \ 1 \ 3 \ 4 \ 3 \ 1 \ 4 \ 2 \ 2 \ 1 \ 4 \ 3$.

Application 6.3. $f: 1, 2, 3, 4, 5 \rightarrow 1, 2, 3, 4, 5$; $N_1 = 5! = 120$ (bijective functions); $C_5^1 = 5$.

$$\begin{aligned} &(C_5^1(1, 2, 3, 4, 5) + C_5^1(1, 2, 3, 5, 4) + C_5^1(1, 2, 4, 3, 5) + C_5^1(1, 2, 4, 5, 3) + C_5^1(1, 2, 5, 3, 4) + \\ &+ C_5^1(1, 2, 5, 4, 3)) + (C_5^1(1, 3, 2, 4, 5) + C_5^1(1, 3, 2, 5, 4) + C_5^1(1, 3, 4, 2, 5) + \\ &+ C_5^1(1, 3, 4, 5, 2) + C_5^1(1, 3, 5, 2, 4) + C_5^1(1, 3, 5, 4, 2)) + (C_5^1(1, 4, 2, 3, 5) + C_5^1(1, 4, 2, 5, 3) + \end{aligned}$$

$$+ C_5^1(1,4,3,2,5) + C_5^1(1,4,3,5,2) + C_5^1(1,4,5,2,3) + C_5^1(1,4,5,3,2) + (C_5^1(1,5,2,3,4) + \\ + C_5^1(1,5,2,4,3) + C_5^1(1,5,3,2,4) + C_5^1(1,5,3,4,2) + C_5^1(1,5,4,2,3) + C_5^1(1,5,4,3,2)) = 120.$$

Because the construction algorithm has direct access facility, we count all bijective functions, but we illustrate only some of them, related with

$$C_5^1(1,2,3,4,5) \text{ and } C_5^1(1,5,4,3,2).$$

| Nr | 1 | 2 | 3 | 4 | 5 | | 116 | 117 | 118 | 119 | 120 | (circuit addresses) |
|---------|---|---|---|---|---|-------|-----|-----|-----|-----|-----|---------------------|
| $f(1)=$ | 1 | 2 | 3 | 4 | 5 | | 1 | 5 | 4 | 3 | 2 | |
| $f(2)=$ | 2 | 3 | 4 | 5 | 1 | | 5 | 4 | 3 | 2 | 1 | |
| $f(3)=$ | 3 | 4 | 5 | 1 | 2 | | 4 | 3 | 2 | 1 | 5 | |
| $f(4)=$ | 4 | 5 | 1 | 2 | 3 | | 3 | 2 | 1 | 5 | 4 | |
| $f(5)=$ | 5 | 1 | 2 | 3 | 4 | | 2 | 1 | 5 | 4 | 3 | |

7. Conclusions

The work is presented at a level suitable for computer programming.

There are two methods to write a mathematical work. We begin by formulating a practical problem and to seek for the mathematical solution (direct problem). Or, we imagine a mathematical theory and look for a practical interpretation (inverse problem). Both methods are good. Our work is based on direct problem. But, now we can mention several new interpretations of discrete functions. For example, if we associate to each digit one color, we can apply the discrete functions in textile industry. Let us say $f: 1, 2, 3 \rightarrow 1$ (red), 2 (green), 3 (yellow).

References

- [1] Oscar Levin (2015). Discrete Mathematics. An Open Introduction. American Institute of Mathematics. Open Textbook Initiative.
- [2] Susanna S. Epp (2010-08-04). Discrete Mathematics with Applications. Thomson Books Code. ISBN 978-0 201-72634-3.
- [3] Tom Jekyns, Ben Stephenson (2010). Fundamentals of Discrete Mathematical Functions. Computer Science, A problem solving primer. Springer.
- [4] Kenneth H. Rosen (2007). Discrete Mathematics and its Applications. McGraw-Hill College. ISBN 978-0-07-288008-3.
- [5] Popa V. Mircea (2005). Matematica Aplicata. Universitatea Lucian Blaga Sibiu.
- [6] Popa V. Mircea (2005). Aranjamente generalizate. Educatia Matematica. Universitatea Lucian Blaga, Sibiu. Vol. 1, Nr. 2, pag. 49-58.
- [7] Andrew Simpson (2002). Discrete Mathematics by Examples. McGraw-Hill Incorporated. ISBN 978-0-07-709840-7.
- [8] Popoviciu Nicolae (1998). Transformata Fourier Rapidă și Teoria Numerelor. Editura Academiei Tehnice Militare, Bucuresti, 1998, 206 de pagini. (Fast Fourier Transform and Number's Theory. Convolution Theory).
- [9] Kenneth H. Rosen (1991). Discrete Mathematics and its Applications. ISBN 978 1260 09.
- [10] Norman L. Biggs (1990). Discrete Mathematics (revised edition).
- [11] Popa V. Mircea (1986). Asupra numerotarii bijectiilor intre doua multimi multiple. Gazeta Matematica, Perfectionare Metodica si Metodologica. Vol. VII, Nr. 2, Bucuresti, pag. 78-81.
- [12] Popa V. Mircea (1980). Unele generalizari in Combinatorica. Buletinul Stiintific al Institutului de Invatamant Superior, Sibiu. Vol. III, pag. 33-30.
- [13] Vermani Lekh R. A Cours in Discrete Mathematical Structures. Imperial College Press.
- [14] Balakrishnan V. K. Introductory Discrete Mathematics. Edition Dover Publications.