Mathematics Letters
2022; 8(2): 22-31
http://www.sciencepublishinggroup.com/j/ml
doi: 10.11648/j.ml.20220802.11
ISSN: 2575-503X (Print); ISSN: 2575-5056 (Online)

Common Fixed Point Theorems for Generalized *R′*-contraction in b-metric Spaces

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To cite this article:
Soressa Wakesa Bekana. Common Fixed Point Theorems for Generalized R′-contraction in b-metric Spaces. *Mathematics Letters*. Vol. 8, No. 2, 2022, pp. 22-31. doi: 10.11648/j.ml.20220802.11

Received: February 26, 2022; **Accepted**: April 8, 2022; **Published**: April 22, 2022

Abstract: In this paper, the researcher presents some common fixed point theorems for self-mappings satisfying generalized R'-contraction in b-metric spaces and obtained a unique common fixed point of a self-mapping satisfying certain contraction in the framework of b- metric spaces. The results presented over her generalize and extend some existing results in the literature. Finally, he illustrate example to support the results.

Keywords: Common Fixed Point, *b*-metric Space, R'-contraction, R'-function, Weakly Compatible Mapping, Simulation Function

1. Introduction

Fixed point theory is a branch of nonlinear analysis that can be applied successfully to a wide range of contexts in social and natural Sciences. Although some results had seen introduced, it is usually considered that this field of study was born in 1922, when Banach presented a celebrated theorem in order to guarantee that a nonlinear operator had generalization, in many different frame works, have been done.

The concept of $b-metric$ space was introduced by Czerwik [4] and formally defined a $b-metric$ space with a view of generalizing the Banach contraction mapping theorem. The well-known Banach contraction principle assures the existence and uniqueness of fixed points of certain self-maps in metric spaces. It is well known that fixed point theory has wide application in applied Science. Banach contraction principle which states that if (X,d) is complete metric space and $f: X \to X$ is a contraction map then f has a unique fixed point, it is a fundamental result in this theory. Due to its importance and simplicity several authors have obtained many interesting extensions and generalization of Banach contraction principle, some generalizations of contraction condition was conducted. This principle can be applied in various fields such us engineering, economics, computer science. Because of its wide applications, several researchers have extended, improved and generalized the result in many directions.

On the other hand side, Bakhtin [3] and Czerwik [4] develop the notion of $b-metric$ space and established some fixed point theorems in $b-metric$ spaces. Subsequently, several results appeared in this direction [5-8, 10]. Recently, Mongkolkeha et al. [9] introduced the notion of a simulation function in the setting of $b-metric$ spaces. In 2002, Aamari and Moutawakil [18] introduced the notation of property $(E.A)$ to prove the existence of common fixed point in metric spaces.

Very recently, Khojasteh et al [1] introduced the notion of simulation function, which was later modified by Roldan Lopez de Hierro et al. in a subtle way [2]. The concept of $b-metric$space was introduced by Bakhtain [3] in 1989, which used it to prove a generalization of the Banach contraction principle in space endowed with such kind of metrics. Since then, this notion has been used by many authors to obtain various fixed point theorems. This direction was the source of several (common) fixed point and coincidence point theorems in various ambient spaces. The concept of compatibility was used by many authors to prove existence theorems in common fixed point theory. The study of common fixed points of weakly compatible mappings in also important. In this work, we define a generalization of $R'-$ contraction in $b-metric$ spaces, called $R'-$contraction, via $R'-$function and prove the existence and uniqueness of common fixed point result for two mapping satisfying weakly compatible condition in the frame work of complete $b-metric$ spaces.

2. Preliminaries

In this section, we recollect some basic definitions, examples and results which are needed in continuance.

Definition 2.1. [4] A $b-metric$ on a set X is a mapping $d: X \times X \to [0,+\infty)$ satisfying the following conditions: for any $x, y, z \in X$,

$$\text{(b1) } d(x,y) = 0 \text{ if and only if } x = y;$$

$$\text{(b2) } d(x,y) = d(y,x);$$

$$\text{(b3) there exists } K \geq 1 \text{ such that } d(x,z) \leq K[d(x,y) + d(y,z)].$$

A pair (X, d) is called a $b-$metric space with parameter K.

It should be noted that the class of $b-metric$ space is effectively larger than that of metric spaces. Indeed, a $b-$metric is metric if and only if $K = 1$.

Example 2.3

1. Let $X = \mathbb{R}$. Define a mappings $d: X \times X \to [0,\infty)$ by

$$d(x,y) = (x-y)^2 \text{ for all } x, y \in X.$$

Then (X,d) *is a* $b-metric$ space with coefficient $K = 2$.

2. Let $X = \{1,2,3\}$. *Define a mapping* $d: X \times X \to [0,\infty)$ *by* $d(1,1) = d(2,2) = d(3,3) = 0, d(1,2) = d(2,1) = 2, d(2,3) = d(3,2) = 1$ *and* $d(1,3) = d(3,1) = 6$. Then (X,d) *is a* $b-metric$ space with coefficient $K = 2$.

In 2015, Khojasteh et al. [1] introduced a simulation function shown below:

Definition 2.4. [1] A simulation function is a mapping $\varsigma: [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfy the following conditions:

$$(\varsigma_1)\ \varsigma(0,0) = 0;$$

$$(\varsigma_2)\ \varsigma(t,s) < s - t, \text{ for all } t, s > 0;$$

$$(\varsigma_3) \text{ if } \{t_n\}, \{s_n\} \text{ are sequence in } [0,\infty) \text{ such that } \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0, \text{ then}$$

$$\lim_{n\to\infty} \varsigma(t_n, s_n) < 0.$$

The class of all simulation functions $\varsigma: [0,\infty) \times [0,\infty) \to \mathbb{R}$ is denoted by $\mathcal{Z}$.

The following are examples of simulation functions given by Khojasteh [1].

Example 2.5.

1. Let $\lambda \in \mathbb{R}$ such that $\lambda < 1$ and define a mapping $\varsigma: [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\varsigma(t,s) = \lambda s - t$ *for all* $s, t \in [0,\infty)$.
2. Define a mapping $\varsigma: [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\varsigma(t,s) = \psi(s) - \phi(t)$ for all $s, t \in [0,\infty)$, where $\psi, \phi: [0,\infty) \to [0,\infty)$ are two continuous fuctions such that $\psi(t) = \phi(t) = 0$ *if and only if* $t = 0$ *and* $\psi(t) < t \leq \phi(t)$ for all $t < 0$, then, $\varsigma \in \mathcal{Z}$.

In 2015, Roldan Lopez de Hierro and shahzad [2] introduced R-function and R-contraction in metric spaces shown below:

Definition 2.6. [2] Let $A \subseteq \mathbb{R}$ be a nonempty subset. A function $\varrho: A \times A \to \mathbb{R}$ is called R-function if it satisfies the following two conditions:

(ϱ_1) *If* $\{a_n\} \subset (0,\infty) \cap A$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$,

$$\text{then } \{a_n\} \to 0.$$

(ϱ_2) *If* $\{a_n\}, \{b_n\} \subset (0,\infty) \cap A$ are two sequences converging to the same limit.

$L \geq 0$ and verifying $L < a_n$ *and* $\varrho(a_n, b_n) > 0$ *for all* $n \in \mathbb{N}$, *then* $L = 0$.

The class of all R-functions $\varrho: A \times A \to \mathbb{R}$ is denoted by R_A. They also consider the following property.

(ϱ_3) *If* $\{a_n\}, \{b_n\} \subset (0,\infty) \cap A$ are two sequence such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) > 0$ *for all* $n \in \mathbb{N}$, *then* $\{a_n\} = 0$. In [2], the authors showed that every simulation function is R-function that satisfies (ϱ_3) but the converse is not true.

In 2018, A. Wiriyapongsanon and N. Phudolsittihiphat [15] introduced $R'-$ contraction and $R'-$ function and prove theorem 2.8 in the frame-work of $b-metric$ spaces shown below:

Definition 2.7. [15] Let (X,d) be a $b-metric$ space with coefficient $K \geq 1$ and let $T: X \to X$ is called R' −contraction if there exists an R' −function $\varrho: [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that:

$$\varrho\big(Kd(Tx,Ty), d(x,y)\big) > 0 \text{ for all } x, y \in X \text{ such that } x \neq y.$$

Theorem 2.8. Let (X,d) be a $b-metric$ space with coefficient $K \geq 1$ and let $T: X \to X$ is called R' −contraction with respect $\varrho \in \mathbb{R}^*$. *If* $\varrho(Kt,s) \leq s - Kt$, *for all* $t, s \in [0,\infty)$, then T has a unique fixed point.

In 2017, Mongkolkeha et al. [9] introduced a simulation function in the frame-work of $b-metric$ spaces shown below:

Definition 2.9. [9] Let K be a given real number such that $K \geq 1$. A K −simulation function is a mapping $\varsigma: [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

$$(\varsigma'_1)\ \varsigma(0,0) = 0;$$

$$(\varsigma'_2)\ \varsigma(Kt,s) < s - Kt, \text{ for all } t, s > 0;$$

(ς'_3) *if* $\{t_n\}, \{s_n\}$ are sequence in $[0,\infty)$ such that $\lim_{n\to\infty} sup\, Kt_n = \lim_{n\to\infty} sup\, s_n$ an $t_n < s_n$ *for all* $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} sup\, \varsigma(Kt_n, s_n) < 0.$$

The class of all $K-$ simulation function $\varsigma: [0,\infty) \times [0,\infty) \to \mathbb{R}$ is denoted by Z^*.

Definition 2.10. [16] Two self-mappings f and g of a metric space (X,d) are said to be weakly compatible if $fu = gu, for\ u \in X\ implies\ fgu = gfu$.

Definition 2.11. [16] Let $f\ and\ g$ be selfmaps on a $b-metric$ space $(X,d). If\ fx = gx = w$ for some $x \in X$, then x is called a coincidence point of $f\ and\ g$ and the set of all coincidence point of $f\ and\ g$ is denoted by $C(f,g)$, and w is called point of coincidence of $f\ and\ g$.

Definition 2.12. [17] A pair (f,g) of self-maps on a $b-metric$ space (X,d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, for\ some\ z\ in\ X$.

Definition 2.13. [17] A pair (f,g) of self-maps on a $b-metric$ space (X,d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, for\ some\ z\ in\ X$.

In 2002, Aamari and Moutawakil [18] introduced the notation of property $(E.A)$ to prove the existence of common fixed point in metric spaces.

Definition 2.13.[18] A pair (f,g) of self- maps on a $b-metric$ space (X,d) is said to be satisfy property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, for\ some\ z\ in\ X$.

Definition 2.14. [15] Let K be a given real number such that $K \geq 1$. $A\ function$ $\varrho: [0,\infty) \times [0,\infty) \to \mathbb{R}\ is\ called\ R' - function$ if it satisfies the following two conditions:

$(\varrho'_1)\ If\ \{a_n\} \subset (0,\infty)$ is a sequence such that $\varrho(Ka_{n+1}, a_n) > 0$ for al $n \in N$,

$$then\ \{a_n\} \to 0.$$

$(\varrho'_2)\ If\ \{a_n\},\{b_n\} \subset (0,\infty)$ are two sequence such that $\lim_{n\to\infty} sup\ Ka_n = \lim_{n\to\infty} sup\ b_n = L \geq 0$ and verifying $L < Ka_n\ and$

$$\varrho(Ka_n, b_n) > 0\ for\ all\ n \in \mathbb{N}, then\ L = 0.$$

The class of all $R'-$functions $\varrho: [0,\infty) \times [0,\infty) \to \mathbb{R}$ is denoted by R^*. They also consider the following property.

$(\varrho'_3)\ If\ \{a_n\},\{b_n\} \subset (0,\infty)$ are two sequence such that $\{b_n\} \to 0$ and $\varrho(Ka_n, b_n) > 0\ for\ all\ n \in \mathbb{N}$, then $\{a_n\} \to 0$.

Lemma 2.15. $Every\ K - simulation\ function\ is\ a\ R' - function\ that\ verifies\ (\varrho'_3)$.

$Proof$: Let K be a given real number such that $K \geq 1\ and\ \varrho: [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a $K - simulation$.

(ϱ'_1) Let $\{a_n\} \subset (0,\infty)$ is a sequence such that $\varrho(Ka_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$.

By condition (ς'_2), $\varrho(Ka_{n+1}, a_n) \leq a_n - Ka_{n+1} \leq a_n - a_{n+1}$.
for all $n \in N$. So $\{a_n\}$ is strictly decreasing sequence of positive real numbers.

Then $\{a_n\}$ is convergent, given $L \geq 0$ such that $\{a_n\} \to L$. We will show that
$L = 0$. By contradiction, assume $L > 0$. Let $t_n = \frac{a_{n+1}}{K}$ and $s_n = a_n$ for all $n \in \mathbb{N}$.

By condition (ς'_3),$0 \leq \lim_{n\to\infty} sup\ \varrho(a_{n+1}, a_n) = \lim_{n\to\infty} sup\ \varrho(Ka_{n+1}, a_n) < 0$,
which is a contradiction. Therefore, $\{a_n\} \to 0$

(ϱ'_2) Let $\{a_n\},\{b_n\} \subset (0,\infty)$ be sequence such that $\lim_{n\to\infty} sup\ Ka_n = \lim_{n\to\infty} sup b_n = L \geq 0$, and satisfying that $L < Ka_n\ and\ \varrho(Ka_n, b_n) > 0$
for all $n \in \mathbb{N}$. We will show that $L = 0$. By contradiction, assume $L > 0$.

By condition (ς'_2),
$0 < \varrho(Ka_n, b_n) \leq b_n - Ka_n$. Then, $a_n \leq Ka_n < b_n$ for all $n \in \mathbb{N}$.

By condition (ς'_3), $0 \leq \lim_{n\to\infty} sup\ \varrho(Ka_n, b_n) < 0$, this is a contradiction.

Therefore, $L = 0$.

(ϱ'_3) Let $\{a_n\},\{b_n\} \subset (0,\infty)$ be sequence such that $\{b_n\} \to 0$ and $\varrho(Ka_n, b_n) > 0$
for all $n \in \mathbb{N}$. Since $\varrho\ is\ a\ k - simulation$ function, $0 < \varrho(Ka_n, b_n) \leq b_n - Ka_n$
for all $n \in \mathbb{N}$. Hence $0 < Ka_n < b_n$ for all $n \in \mathbb{N}$, this implies that, $\{Ka_n\} \to 0$.

Since, $K \geq 1$, $\{a_n\} \to 0$.

Lemma 2.16. If $\varrho(Kt, s) \leq s - kt\ for\ all\ t, s \in (0.\infty), then\ (\varrho'_3)$ holds.

$Proof$: Assume that $\{a_n\},\{b_n\} \subset (0,\infty)$ are two sequence such that $\{b_n\} \to 0$ and $\varrho(Ka_n, b_n) > 0\ for\ all\ n \in \mathbb{N}$. Since $a_n, b_n \in (0,\infty)$, then $0 < \varrho(Ka_n, b_n) \leq b_n - Ka_n$ for all $n \in \mathbb{N}$. As a consequence, $0 < Ka_n < b_n$ for all $n \in \mathbb{N}$, which means that $\{a_n\} \to 0$.

Definition 2.17. [19] Let (X,d) be a $b - metric$ space. Then a subset $Y \subset X$ is called closed if and only if each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$ $(i, e\ \overline{Y} = Y)$.

Remark 2.18. [19].

1. The $b - metric$ space (X,d) is complete if every Cauchy sequence in X is converges.
2. In the $b - metric$ space (X,d), the sequence $\{x_n\}$ in X is called convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. in this case, write $\lim_{n\to\infty} x_n = x$.
3. The sequence $\{x_n\}$ is called Cauchy sequence if and only if $d(x_n, x_m) \to 0, as\ n, m \to \infty$.

3. Main Result

Definition 3.1. Let $(X,d)\ be\ a\ b - metric$ space with coefficient, $K \geq 1$ and let $f,g: X \to X$ are mappings. We will say that $f\ and\ g\ are\ R' - contractions$ if there exists a $R' - function\ \varrho: [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

$$\varrho\big(Kd(fx,fy), M(x,y)\big) > 0, for\ all\ x,y \in X\ such\ that\ x \neq y$$

Where,

$$M(x,y) = \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2K}\right\}$$

Proposition 3.2. Let (X,d) be a $b-metric$ space with coefficient $K \geq 1$ and $f,g: X \to X$ be two self-maps. Assume that f,g is a generalized R' – contraction pair maps. Then u is a fixed point f if and only if u is a fixed point of g. Moreover, in that case u is a unique.

Proof: let u be a fixed point of f. i.e $fu = u$. Suppose $gu \neq u$.

We consider,

$$\varrho\big(Kd(fu,gu), M(u,u)\big) > 0, \quad (1)$$

Where

$$M(u,u) = \max\left\{d(u,u), d(u,fu), d(u,gu), \frac{d(u,gu)+d(u,fu)}{2K}\right\},$$

$$= \max\left\{0,0,d(u,gu), \frac{d(u,gu)}{2K}\right\},$$

$$= d(u,gu)$$

Now using the value of $M(u,u)$ in (1), we get

$$0 \leq \varrho\big(Kd(fu,gu), M(u,u)\big) = \varrho\big(Kd(u,gu), d(u,gu)\big) < d(u,gu) - Kd(u,gu) \leq 0$$

a contradiction. Hence $gu = u$, so that u is a common fixed point of f and g. Similarly, it is easy to see that if u is a fixed point of g then u is fixed point of f also.

Then,

$$\big(Kd(fu,ft), M(u,t)\big) = \varrho\big(Kd(fu,ft), d(z,v)\big) > 0.$$

Suppose u and v are common fixed points of f and g with $u \neq v$

By $R' - contraction$,

$$\varrho\big(Kd(u,v), M(u,v)\big) > 0$$

Where,

$$M(u,v) = \max\left\{d(u,v), d(u,fu), d(v,gv), \frac{d(u,gv)+d(v,gu)}{2K}\right\},$$

$$= \max\left\{d(u,v), 0,0, \frac{d(z,v)+d(v,z)}{K}\right\},$$

$$= \max\left\{d(z,v), 0,0, \frac{d(u,v)}{K}\right\},$$

$$= d(u,v)$$

Now using the value of $M(u,u)$ in (1), we get

$$0 \leq \varrho\big(Kd(fu,gv), M(u,v)\big) = \varrho(Kd(u,v), d(u,v) < d(u,v) - Kd(u,v) \leq 0$$

a contradiction. Hence $u = v$. Therefore the proposition follows.

Theorem 3.3. Let (X,d) $be\ a\ complete\ b-metric$ space with coefficient $K \geq 1$ and $f,g: X \to X$ be self-maps of X, with $fX \subset gX$. Let $f\ and\ g\ be\ R' - contraction$ with respect to $\varrho \in R^*$. If $\varrho(Kt,s) \leq s - kt\ for\ all\ t,s \in X$, then for any $x_0 \in X$, the Picard iterates $\{y_n\}$ defined by

$y_n = fx_n = gx_{n+1}\ for\ all\ n = 0,1,2,\ldots$ is a Cauchy sequence in X.

$Proof$: Let $x_0 \in X$. Since $fX \subset gX$ there exists $x_1 \in X$ such that $y_0 = fx_0 = gx_1$. Further corresponding to x_1, there exists $x_2 \in X$ such that $y_1 = fx_1 = gx_2$. On continuing process, inductively we obtain a sequence $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}, for\ all\ n = 0,1,2,3, \quad (2)$$

Now, we consider the following cases.

$Case\ (i)$: Suppose $y_n = y_{n+1}$ for some $n \in \mathbb{N}$.

By $R' - contraction$, we have

$$\varrho(Kd(fx_{n+1}, fx_{n+2}), M(x_{n+1}, x_{n+2})) > 0$$

Where,

$$M(x_{n+1},x_{n+2}) = \max\{d(gx_{n+1},gx_{n+2}), d(gx_{n+1},fx_{n+1}), d(gx_{n+2},fx_{n+2}), \frac{d(gx_{n+1},fx_{n+2})+d(gx_{n+2},fx_{n+1})}{2K}\}$$

$$= \max\left\{d(y_n,y_{n+1}), d(y_n,y_{n+1}), d(y_{n+1},y_{n+2}), \frac{d(y_n,y_{n+2})+d(y_{n+1},y_{n+1})}{2K}\right\}$$

$$\leq \max\left\{d(y_n,y_{n+1}), d(y_n,y_{n+1}), d(y_{n+1},y_{n+2}), \frac{K\big(d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})\big)+0}{2K}\right\}$$

$$= \max\{d(y_n,y_{n+1}), d(y_{n+1},y_{n+2})\}$$

$$M(x_{n+1},x_{n+2}) = \max\{d(y_n,y_{n+1}), d(y_{n+1},y_{n+2})\}$$

Suppose $d(y_n,y_{n+1}) \leq d(y_{n+1},y_{n+2})$, then

$$M(x_{n+1},x_{n+2}) = \max\{d(y_n,y_{n+1}), d(y_{n+1},y_{n+2})\} = d(y_{n+1},y_{n+2})$$

Hence, $\varrho\big(Kd(fx_{n+1},fx_{n+2}), M(x_{n+1},x_{n+2})\big) = \varrho(Kd(y_{n+1},y_{n+2}), d(y_{n+1},y_{n+2})) > o$

$$= d(y_{n+1},y_{n+2}) - Kd(y_{n+1},y_{n+2}) \leq 0$$

a contradiction.

Therefore, $d(y_n, y_{n+1}) \geq d(y_{n+1}, y_{n+2})$,

Now, $\varrho\big(Kd(y_{n+1}, y_{n+2}), d(y_n, y_{n+1})\big) > o$,

from condition (ϱ_1'), $d(y_n, y_{n+1}) = 0$ and let $a_n = d(y_{n+1}, y_{n+2})$ and $b_n = d(y_n, y_{n+1})$ are the two sequence and $\{b_n\} \to 0$, then by condition (ϱ_3'), we have $\{a_n\} \to 0$, that is $d(y_{n+1}, y_{n+2}) = 0$.

Therefore, $y_{n+2} = y_{n+1} = y_n$.

Similarly, we can show that $y_{n+3} = y_{n+2} = y_{n+1} = y_n$.

This implies that $y_m = y_n \; for \; all \; m > n, so \; that \; \{y_m\}_{m>n}$ is constant sequence.

Hence $\{y_m\}$ is a Cauchy sequence.

$Case \; (ii)$: $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$.

By $R' - contraction$

$$\varrho\big(Kd(fx_{n+1}, fx_{n+2}), M(x_{n+1}, x_{n+2})\big) > 0,$$

Where,

$$M(x_{n+1}, x_{n+2}) = \max\{d(gx_{n+1}, gx_{n+2}), d(gx_{n+1}, fx_{n+1}), d(gx_{n+2}, fx_{n+2}), \tfrac{d(gx_{n+1}, fx_{n+2}) + d(gx_{n+2}, fx_{n+1})}{2K}\}$$

$$= \max\left\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \tfrac{d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})}{2K}\right\}$$

$$= \max\left\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \tfrac{d(y_n, y_{n+2})}{2K}\right\}$$

$$\leq \max\left\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \tfrac{d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})}{2}\right\}$$

$$= \max\{d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}$$

$$= \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}$$

Hence, $M(x_{n+1}, x_{n+2}) = \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}$.

Suppose that $d(y_n, y_{n+1}) \leq d(y_{n+1}, y_{n+2})$, for some $n \in \mathbb{N}$.

Then we have,

$$M(x_{n+1}, x_{n+2}) = \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\} = d(y_{n+1}, y_{n+2})$$

Hence,

$$0 < \varrho\big(Kd(fx_{n+1}, fx_{n+2}), M(x_{n+1}, x_{n+2})\big) = \varrho\big(Kd(fx_{n+1}, fx_{n+2}), d(y_{n+1}, y_{n+2})\big)$$

$$= \varrho\left(Kd\big((y_{n+1}, y_{n+2})\big), d(y_{n+1}, y_{n+2})\right) < d(y_{n+1}, y_{n+2}) - Kd(y_{n+1}, y_{n+2}) \leq 0$$

a contradiction.

Hence, $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, for all $n \in \mathbb{N}$.

Therefore, $\{d(y_n, y_{n+1})\}$ is decreasing and bounded below. Thus there exist $r \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$.

Suppose that $r > 0$. Now, using condition of (ς'_3), with $a_n = d(y_{n+1}, y_{n+2})$ and $b_n = d(y_n, y_{n+1})$, we have

$$0 \leq \lim_{n\to\infty} sup\, \varrho\big(Kd(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2})\big) < 0,$$

a contradiction. Therefore, $r = 0$. That is

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence.

Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$, such that

$$d(y_{nk}, y_{mk}) > \epsilon \; and \; d(y_{nk}, y_{mk-1}) \leq \epsilon \; for \; all \; m_k > n_k \geq k \tag{3}$$

where $\{m_k\} \; and \; \{n_k\}$ are sequence of positive integer.

Consider,

$$\epsilon < d(y_{nk}, y_{mk}) \leq K(d\big(y_{nk}, y_{mk-1}\big) + d(y_{mk-1}, y_{mk})\big), \text{ for all } k \in \mathbb{N}.$$

Taking limit superior k to infinity

$$\epsilon \leq \lim_{n\to\infty} sup d(y_{nk}, y_{mk}) \leq K\epsilon \tag{4}$$

Since $d(y_{nk-1}, y_{mk-1}) \leq K(d(y_{nk-1}, y_{nk}) + d(y_{nk}, y_{mk-1}))$, for all $k \in \mathbb{N}$.
Taking limit superior k to infinity

$$\lim_{n\to\infty} sup d(y_{nk-1}, y_{mk-1}) \leq K\epsilon \tag{5}$$

If $d\left(y_{nk_0-1}, y_{mk_0-1}\right) = 0$, for some $k_0 \in \mathbb{N}$, $then, y_{nk0} = y_{mk_0}$, which contradict to (4). Therefore, $y_{nk-1} \neq y_{mk-1}$ for all $k \in \mathbb{N}$.

By $R' - contraction$ we have,

$$\varrho(Kd(fx_{nk}, gx_{mk}), M(x_{nk}, x_{mk})) > 0,$$

Where,

$$M(x_{nk}, x_{mk}) = \max\{d(gx_{nk}, gx_{mk}), d(gx_{nk}, fx_{nk}), d(gx_{mk}, fx_{mk}), \tfrac{d(gx_{nk}, fx_{mk}) + d(gx_{mk}, fx_{nk})}{2K}\}$$

$$= \max\{d(y_{nk-1}, y_{mk-1}), d(y_{nk}, y_{nk}), d(y_{mk-1}, y_{mk}), \tfrac{d(y_{nk-1}, y_{mk}) + d(y_{mk-1}, y_{nk})}{2K}\},$$

$$= \max\{d(y_{nk-1}, y_{mk-1}), d(y_{nk}, y_{nk}), d(y_{mk-1}, y_{mk})\}$$

$$= \max\{d(y_{nk-1}, y_{mk-1}), d(y_{mk-1}, y_{mk})\}\ M(x_{nk}, x_{mk}) = \max\{d(y_{nk-1}, y_{mk-1}), d(y_{mk-1}, y_{mk})\},$$

Suppose that $d(y_{nk-1}, y_{mk-1}) \leq d(y_{mk-1}, y_{mk})$, then

$$M(x_{nk}, x_{mk}) = \max\{d(y_{nk-1}, y_{mk-1}), d(y_{mk-1}, y_{mk})\} = d(y_{mk-1}, y_{mk}).$$

Hence,

$$0 < \varrho(Kd(fx_{nk}, gx_{mk}), M(x_{nk}, x_{mk})) = \varrho\left(Kd(y_{nk}, y_{mk}), d(y_{mk-1}, y_{mk})\right) \leq d(y_{mk-1}, y_{mk}) - Kd(y_{nk}, y_{mk}) \leq 0,$$

a contradiction.

Therefore, $(y_{nk-1}, y_{mk-1}) \geq d(y_{mk-1}, y_{mk})$.
Then,

$$0 < \varrho(Kd(y_{nk}, y_{mk}), M(x_{nk}, x_{mk}) = \varrho(Kd(y_{nk}, y_{mk}), d(y_{nk-1}, y_{mk-1})$$

$$= \varrho(Kd(y_{nk}, y_{mk}), d(y_{nk-1}, y_{mk-1}) \leq d(y_{nk-1}, y_{mk-1}) - Kd(y_{nk}, y_{mk})$$

This implies that

$$Kd(y_{nk}, y_{mk}) < d(y_{nk-1}, y_{mk-1}), \text{ for all } k \in \mathbb{N}. \tag{6}$$

Now, by $(3), (4)\ and\ (5)$

$$K\epsilon \leq \lim_{n\to\infty} \sup kd(y_{nk}, y_{mk}) \leq \lim_{n\to\infty} sup d(y_{nk-1}, y_{mk-1}) \leq K\epsilon$$

That is $\lim_{n\to\infty} \sup kd(y_{nk}, y_{mk}) = \lim_{n\to\infty} sup d(y_{nk-1}, y_{mk-1}) = K\epsilon$
Since, $K\epsilon < Kd(y_{nk}, y_{mk})$, for all $k \in \mathbb{N}$ and by condition $({\varrho'}_2)$, $K\epsilon = 0$,
That is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence.

Theorem 3.4. In addition to the hypothesis of theorem 2.3 on f and g, if either $gX\ or\ fX$ is complete, then for any $x_0 \in X$, the Picard iterates $\{y_n\}$ defined by (2) converges to z in X and z is a unique point of coincidence of f and g.

$Proof$: By theorem 3.3 the sequence $\{y_n\}$ is Cauchy in X. Since gX is closed, there exists a point $z \in gX$ such that $\lim_{n\to\infty} y_n = z$. Hence there exists $u \in X$ such that $gu = z$.

Now, we show that $gu = fu$.

Suppose, $gu \neq fu$. By $R' - contraction$ we have,

$$\varrho\left(Kd(gx_{n+1}, fu), M(x_{n+1}, u)\right) > 0$$

Where,

$$d(gu, fu) \leq M(x_{n+1}, u) = \max\{d(gx_{n+1}, gu), d(gx_{n+1}, fx_{n+1}), d(gu, fu), \tfrac{d(gx_{n+1}, fu) + d(gu, fx_{n+1})}{2K}\}$$

$$\leq \max\{d(gx_{n+1},gu),d(gx_{n+1},fx_{n+1}),d(gu,fu),\frac{K(d(gx_{n+1},gu)+Kd(gu,fu)+d(gu,fx_{n+1}))}{2K}\}$$

On taking limit as $n \to \infty$ in the above in equality, we have

$$d(gu,fu) \leq \lim_{n\to\infty} M(x_{n+1},u) = \max\{0,0,d(gu,fu),\frac{d(gu,fu)}{2}\} = d(gu,fu)$$

Therefore, $\lim_{n\to\infty} M(x_{n+1},u) = d(gu,fu)$
Hence, $0 < \varrho\big(Kd(gx_{n+1},fu),M(x_{n+1},u)\big) \leq M(x_{n+1},u) - Kd(gx_{n+1},fu)$
This implies that

$$Kd(gx_{n+1},fu) < M(x_{n+1},u) \tag{7}$$

From triangular inequality we have

$$d(gu,fu) \leq Kd(gu,gx_{n+1}) + Kd(gx_{n+1},fu) \leq Kd(gu,gx_{n+1}) + M(x_{n+1},u)$$

On taking limit as $n \to \infty$, we have

$$d(gu,fu) \leq K\lim_{n\to\infty} d(gx_{n+1},fu) \leq d(gu,fu)$$

Therefore, $K\lim_{n\to\infty} d(gx_{n+1},fu) = d(gu,fu)$
From (7), triangular inequality above and by condition (ϱ_2') $d(gu,fu) = 0$, that is
Contradiction. Therefore,
$gu = fu = z, u$ is a coincidence point of f and g and z is a point of coincidence of f and g.
Now, we show that a point of coincidence of f and g is unique. Suppose for some $t \in X, f(t) = g(t) = v, with\ v \neq z.$
By $R' - contraction$,

$$\varrho\big(Kd(fu,ft),M(u,t)\big) > 0$$

Where,

$$M(u,t) = \max\left\{d(gu,gt),d(gu,fu),d(gt,ft),\frac{d(gu,ft)+d(gt,fu)}{2K}\right\},$$

$$= \max\left\{d(z,v),d(z,z),d(v,v),\frac{d(z,v)+d(v,z)}{2K}\right\},$$

$$= \max\left\{d(z,v),0,0,\frac{d(z,v)}{K}\right\},$$

$$= d(z,v)$$

Then, $\varrho\big(Kd(fu,ft),M(u,t)\big) = \varrho\big(Kd(fu,ft),d(z,v)\big) > 0$
Again by condition $\left(\varrho'_1\right)$, $d(z,v) = 0$, this implies that $z = v$.
Therefore, f and g have a unique point of coincidence in X.

Theorem 3.5. Under the assumption of theorem 3.4, if the pair (f,g) is weakly compatible self –maps then f and g have a unique common fixed point.

$Proof$: By theorem 3.4, z is a point of coincidence of f and g. Since $z \in gX$ there exists $u \in X$ such that $fu = gu = z$. And also since the pair (f,g) is weakly compatible $fgu = gfu$ implies that, $fz = gz$.
Now, we claim that z is a common fixed point of f and g.
Suppose that $fz \neq z$. Then by $R' - contraction$,

$$\varrho(Kd(fz,fu),M(z,u)) > 0$$

Where,

$$M(z,u) = \max\left\{d(gz,gu),d(gz,fz),d(gu,fu),\frac{d(gz,fu)+d(gu,fz)}{2K}\right\}$$

$$= \max\left\{d(fz,gu),d(fz,gz),d(gu,fu),\frac{d(fz,fu)+d(gu,fz)}{2K}\right\}$$

$$= \max\left\{d(fz,z),d(fz,fz),d(z,z),\frac{d(fz,z)+d(z,fz)}{2K}\right\}$$

$$= \max\{d(fz,z),0,0\}$$

$$= d(fz,z)$$

Then,
$\varrho(Kd(fz,fu),M(z,u)) = \varrho(Kd(fz,fu),d(fz,z)) > 0$
From condition (ϱ'_1), then $d(fz,z) = 0$, which implies that $fz = z$, which is a contradiction. Therefore, $gz = fz = z, z$ is a common fixed point of f and g.

Now, show that the common fixed point of f and g is unique. To show this, let z and w be two common fixed points of f and g, with $z \neq w$. Then, we have
By R'-contraction

$$\varrho(Kd(fz,fw),M(z,w)) > 0$$

Where,

$$M(z,w) = \max\left\{d(gz,gw), d(gz,fz), d(gw,fw), \frac{d(gz,fw)+d(gw,fz)}{2K}\right\}$$
$$= \max\left\{d(z,w), d(z,z), d(w,w), \frac{d(z,w)+d(w,z)}{2K}\right\}$$
$$= \max\left\{d(z,w), 0,0, \frac{d(z,w)}{K}\right\}$$
$$= d(z,w)$$

Then, $\varrho(Kd(fz,fw),M(z,w)) = \varrho(Kd(fz,fw),d(z,w)) > 0$
From (ϱ'_1), $d(z,w) = 0$, which implies that, $z = w$.
Therefore, z is a $unique$ $common$ $fixed$ $point$ of f and g.

Corollary 3.6. Let (X,d) be a complete $b-metric$ space with a coefficient $K \geq 1$, and $f,g: X \to X$ are weakly compatible self maps satisfying $fX \subset gX$. We will say that f and g are R' −contraction with respect $\varrho \in \mathbb{R}^*$. If $\varrho(Kt,s) \leq s - Kt$ for all $t,s \in (0,\infty)$ and if

$$\varrho\big(Kd(fx,fy),M(x,y)\big) > 0$$

Where, $M(x,y) = \max\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2K}\}$
Then f and g have a unique fixed point.
Proof: Follows from theorem 3.3 and 3.4 by taking $g = I, I$ is identity map of X.

Corollary 3.7. Let (X,d) be a complete $b-metric$ space with coefficient $K \geq 1$ and let $f,g: X \to X$ are weakly compatible self maps satisfying $fX \subset gX$ and either gX or fX is complete. Suppose that there exists alower-semi continuous function $\varphi: [0,\infty) \to [0,\infty)$ with $\varphi^{-1}(0) = 0$ such that

$$d(fx,gy) \leq M(x,y) - \varphi(M(x,y)), \text{ for all } x,y \in X$$

Where,

$$M(x,y) = \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2K}\right\}$$

then f and g have a unique common fixed point.
Proof: The result follows from the theorem 3.3-3.5 by taking as simulation function.

$$\varsigma(t,s) = s - \varphi(s) - t \; for \; all \; t,s \geq 0$$

Example 3.8. Let $X = [0,1]$ and $d(x,y) = (x-y)^2$ for all $x,y \in X, then$ (X,d) is a complete $b-metric$ space with coefficient $K = 2$, and we define $f,g: X \to X$ by

$$fx = \begin{cases} \frac{1}{2}, x \in (0,\frac{2}{5}) \\ \frac{1}{2} - \frac{x}{4}, x \in [\frac{2}{5},1] \end{cases}$$

$$gx = \begin{cases} 1, x \in (0,\frac{2}{5}) \\ \frac{3}{5} - \frac{x}{2}, x \in [\frac{2}{5},1] \end{cases}$$

Clearly, $fX \subset gX$ and gX is complete. Since $x = \frac{2}{5}$ is the only coincidence point of $f$and g. We have also $fg\left(\frac{2}{5}\right) = gf\left(\frac{2}{5}\right)$ whenever, $f\left(\frac{2}{5}\right) = g(\frac{2}{5})$. Hence f and g are weakly compatible. Since there is a sequence $x_n = \frac{2}{5} + \frac{1}{n}, n \geq 1$ with

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{2}{5}$. Now we define $\varrho: [0,\infty)\times[0,\infty)\to\mathbb{R}\; by\; \varrho(Kt,s) \le s - Kt, for\; all\; t,s\in X, then\; \varrho \in R^*$.

Now we verify the R'-contraction, let us consider the following cases:

$$Case\ (i),: x\in\left(0,\frac{2}{5}\right)\ and\ y\in[\frac{2}{5},1]$$

In this case $fx = \frac{1}{2}, fy = \frac{1}{2}-\frac{y}{4}, gx = 1\ and\ gy = \frac{3}{5}-\frac{y}{2}$

$$0 < \varrho\left(Kd(fx,fy), \max\left\{d(gx.gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2K}\right\}\right),$$

$$= \varrho(2\left(\frac{1}{2}-\frac{1}{2}+\frac{y}{4}\right)^2, \max\left\{(1-\frac{3}{5}+\frac{y}{2})^2, (1-\frac{1}{2})^2, (\frac{3}{5}+\frac{y}{4})^2, \frac{(1-\frac{1}{2}+\frac{y}{4})^2+(\frac{3}{5}-\frac{y}{2}-\frac{1}{2})^2}{4}\right\})$$

$$= \varrho(2\left(\frac{Y}{4}\right)^2, \max\left\{\left(\frac{2}{3}+\frac{y}{2}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{1}{10}-\frac{y}{4}\right)^2, \frac{\left(\frac{1}{2}+\frac{y}{4}\right)^2+\left(\frac{1}{10}-\frac{y}{2}\right)^2}{4}\right\}\right)$$

$$= \varrho(2(\frac{y}{4})^2, (\frac{1}{2})^2),$$

$$\varrho(2(\frac{y}{4})^2, (\frac{1}{2})^2) \le \left(\frac{1}{2}\right)^2 - 2(\frac{y}{4})^2 > 0, \text{ for all } x,y\in[0,1]$$

$$Case\ (ii),: x\in\left[\frac{2}{5},1\right]\ and\ y\in(0,\frac{2}{5})$$

In this case, $fx = \frac{1}{2}-\frac{x}{4}, fy = \frac{1}{2}, gx = \frac{3}{5}-\frac{x}{2}\ and\ gy = 1$

$$0 < \varrho\left(Kd(fx,fy), \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2K}\right\}\right)$$

$$= \varrho\left(2\left(\frac{1}{2}-\frac{x}{4}-\frac{1}{2}\right)^2, \max\left\{\left(\frac{3}{5}-\frac{x}{2}-1\right)^2, \left(\frac{3}{5}-\frac{x}{2}-\frac{1}{2}+\frac{x}{4}\right)^2, \left(1-\frac{1}{2}\right)^2, \frac{\left(\frac{3}{5}-\frac{x}{2}-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}+\frac{x}{4}\right)^2}{4}\right\}\right)$$

$$= \varrho\left(2\left(\frac{-x}{4}\right)^2, \max\left\{\left(\frac{-2}{5}-\frac{x}{2}\right)^2, \left(\frac{1}{10}-\frac{x}{4}\right)^2, \left(\frac{1}{2}\right)^2, \frac{\left(\frac{1}{10}-\frac{x}{2}\right)^2+\left(\frac{1}{2}+\frac{x}{4}\right)^2}{4}\right\}\right)$$

$$= \varrho\left(2\left(\frac{x^2}{16}\right), \frac{1}{4}\right),$$

$$\varrho(2\left(\frac{x^2}{16}\right), \frac{1}{4}) \le \frac{1}{4} - 2\left(\frac{x^2}{16}\right) > 0, for\ all\ x,y\in[0,1]$$

$Case\ (iii),: x,y\in(0,\frac{2}{5})$. In this case, $fx = \frac{1}{2}, fy = \frac{1}{2}, gx = 1\ and\ gy = 1$

$$0 < \varrho\left(Kd(fx,fy), \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2K}\right\}\right)$$

$$= \varrho\left(2\left(\frac{1}{2}-\frac{1}{2}\right)^2, \max\left\{(1-1)^2, \left(1-\frac{1}{2}\right)^2, \left(1-\frac{1}{2}\right)^2, \frac{\left(1-\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2}{4}\right\}\right)$$

$$= \varrho\left(2(0), \frac{1}{2}\right),$$

$$\varrho\left(2(0), \frac{1}{2}\right) \le \frac{1}{2} - 0 > 0,$$

It also holds true for $x,y\in\left[\frac{2}{5},1\right]$.

Hence from all the above cases f and g satisfy the $R'-$ contraction. Therefore, $f\ and\ g$ satisfy all the hypothesis of theorem 3.3, 3.4 and 3.5 and they have a unique common fixed point $x = \frac{2}{5}$.

4. Conclusion

In this paper, we introduced generalized R'-contraction via R'-function and obtain common fixed point theorems in the framework of $b-metric$ spaces. Further we provided example that elaborated the use ability of the results. The result on this thesis generalized and extended several common fixed point results in the literature.

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