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Common Fixed Point Theorem for Four Self-Maps Satisfying (CLR_{ST}) Property for Generalized (ψ, φ) –Weakly Contraction in B-metric Space

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Abstract: This paper is devoted to obtain fixed point results. Fixed point theory is very wide field which have many application in different areas. The concept of altering distance to find fixed point results have been explored by many authors. In the present paper the author state altering distance function and ultra-altering distance function and the coincidence point for two self-mappings that satisfy the (CLR_{ST}) - property with the help of altering distance function and ultra-altering distance function in the context of b-metric spaces and achieve a unique common fixed point for two weakly compatible pairs. Many discoveries can also be derived from these main results in the framework of metric spaces.

Keywords: B-metric Spaces, Coincidence Point, Weakly Compatible, Ultra-Altering Distance Function, Common Fixed Point

1. Introduction and Preliminaries

Fixed point theory is a branch of nonlinear analysis that can be applied successfully to a wide range of contexts in social and natural Sciences. IT is one of the most acclaimed scientific considerations with application in many disciplines, particularly in engineering, physics, economics and medical science etc. This theory is also beneficial for solving varies kind of integral and differential equations. Although some results had seen introduced, it is usually considered that this field of study was born in 1922, when Banach presented a celebrated theorem in order to guarantee that a nonlinear operator had generalization, in many different frame works, have been done.

On the otherhand side, Bakhtin [3] and Czerwik [4] develop the notion of $b-metric$ space and established some fixed point theorems in $b-metric$ spaces. Subsequently, several results appeared in this direction [11, 12, 13, 14, 15]. The concept of $b-metric$ space was introduced by Czerwik [4] and formally defined a $b-metric$ space with a view of generalizing the Banach contraction mapping theorem. The well-known Banach contraction principle assures the existence and uniqueness of fixed points of certain self-maps in metric spaces. It is well known that fixed point theory has wide application in applied Science. Banach contraction principle which states that if (X, d) is complete metric space and $f: X \to X$ is a contraction map then f has a unique fixed point, it is a fundamental result in this theory. Due to its importance and simplicity several authors have obtained many interesting extensions and generalization of Banach contraction principle, some generalizations of contraction condition was conducted. Because of its wide applications, several researchers have extended, improved and generalized the result in many directions.

Definition 1.1 [4] let X be a non empty set and $s \geq 1$ be given real number. A function $d: X \times X \to [0, +\infty)$ is a $b-metric$ if for any $p, q, r \in X$, the following conditions are satisfied

$$\text{(b1) } \mathrm{d}(\mathrm{p}, \mathrm{q}) = 0 \text{ if and only if } \mathrm{p} = \mathrm{q}$$

$$\text{(b2) } \mathrm{d}(\mathrm{p}, \mathrm{q}) = \mathrm{d}(\mathrm{q}, \mathrm{p});$$

$$\text{(b3) } \mathrm{d}(\mathrm{p}, \mathrm{q}) \leq \mathrm{s}[\mathrm{d}(\mathrm{p}, \mathrm{r}) + \mathrm{d}(\mathrm{r}, \mathrm{q})].$$

A pair (X, d) is called a $\mathrm{b}-\mathrm{metric}$ space with parameter s. It should be noted that the class of $b-metric$ space is effectively larger than that of metric spaces. Indeed, a $\mathrm{b}-$metric is metric if and only if $\mathrm{s} = 1$.

Example 1.2

1) Let $X = \mathbb{R}$. Define a mappings $d: X \times X \to [0, \infty)$ by

$$\mathrm{d(p,q)} = (p-q)^2 \; for\; all\; p, q \in X.$$

Then (X,d) $is\ a\ b-metric$ space with coefficient $s = 2$.

2) let $X = \{1,2,3\}. Define\ a\ mapping\ d: X \times X \to [0,\infty)\ by\ d(1,1) = d(2,2) = d(3,3) = 0, d(1,2) = d(2,1) = 2, d(2,3) = d(3,2) = 1\ and\ d(1,3) = d(3,1) = 6$. Then $(X,d)\ is\ a\ b-metric$ space with coefficient $s = 2$.

In 2015, Ozterk et al [5] give the notion of property $b-(E.A)$ property in b-metric space as follows:

Definition 1.3. [5] A pair (S,T) of self- maps on a $b-metric$ space (X,d) is said to be satisfy $b-(E.A)$ property if there exists a sequence $\{y_n\}$ in X such that, $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = p, for\ some\ p\ in\ X$.

Definition 1.4. [8] Let $\{y_n\}$ be a sequence in a $b-metric$ space (X,d).

a) $\{y_n\}$ is called $b-\ convergent$ if and only if there is $p \in X$ such that $d(y_n, p) \to 0\ as\ n, m \to 0$.

b) $\{y_n\}$ is called $b-\ Cauchy$ sequence if and only if $d(y_n, y_m) \to 0\ as\ n, m \to 0$.

Proposition 1.5 [3] In $b-metric$ space (X,d) the following assertions hold:

1) A $b-convergen$t sequence has a unique limit.

2) Each $b-\ convergent$ sequence is b-Cauchy.

3) In general a $b-metric$ space is not continuous.

Definition 1.6 [1] A function $\psi: [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

1) ψ is non-decreasing and continuous,

2) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.7 [7] An ultra-altering distance function is continuous, non-decreasing mapping $\varphi: [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0 or\ t = 0$ and $\varphi(0) \geq 0$.

Definition 1.8 [6] Let A and B be given self-mappings on a non-void set X. The pair (A,B) is said to be weakly compatible if A and B commute at their coincidence points (i.e. $ABp = BAp$ whenever $Ap = Bp$).

In 2011, Sintunavarat et al. [2] introduce the notion of $CLR_{ST} - property$ in the context of metric space as follows.

Definition 1.9 [2] Two self-mapping A and B of a metric space (X,d) are said to satisfy (CLR_A) property if there exists a sequence $\{y_n\}$ in X such that,

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} By_n = Ap$$

for some p in X.

In 2009, Zhang and Song [9] used generalized φ −weak contraction which is defined for two mappings and gave conditions for existence of a common fixed point.

Theorem 1.10 [9] (see [7]). Let (X, d) be complete metric space, and $T, S: X \to X$ two mappings such that for all $x, y \in X$,

$$d(Tx, Sy) \leq M(x,y) - \varphi(M(x,y))$$

Where $\varphi: [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\varphi(t) > 0$ for $t \in (0,\infty)$ and $\varphi(0) = 0$.

$$M(x,y) = \max\{d(x,y), d(Tx,x), d(Sy,y), \tfrac{[d(y,Tx)+d(x,Sy)}{2}\}$$

then there exists the unique point $u \in x\ such\ that\ u = Tu = Su$.

1n 2020, Sahi Arora et al [10] define C-class function and prove the below theorem,

Definition 1.11 [10] A mapping $F: [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called C-class function it is continuous and fulfill the conditions:

1) $F(s,t) \leq s$

2) $F(s,t) = s$, implies that ether $s = 0$ or $t = 0$, for all $s, t \in [0,\infty)$.

Theorem 1.12 (see [10]) Let (X,d) be a $b-metric$ spaces and $P, Q, S, T: X \to X$ be mapping with $P(X) \subseteq T(X)$ and $Q(X) \subseteq S(X)$, Such that:

$\psi(d(Px,Qy)) \leq F(\psi(M(x,y)), \varphi(M(x,y)))$, for all $x, y \in X$.

Where,

$$M(x,y) = \max\{d(Sx,Ty), d(Px,Sx), d(Qy,Ty), \tfrac{[d(Px,Ty)+d(Sx,Qy)}{2s}\}$$

then P, Q, S, T has a unique common fixed point in X.

2. Main Result

In the following theorem, we show the existence and uniqueness of common fixed point for four self-maps.

Theorem 2.1. Let (X,d) be a $b-metric$ space with $s \geq 1$ and A, B, S, T: $X \to X$ be self-maps with $A(X) \subseteq T(X)\ and\ B(X) \subseteq S(X)$ such that

$$\psi(d(Ax,By)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y)), \text{ for each} x, y \in X: \tag{1}$$

Where,

$$M_s(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \tfrac{d(Ax,Ty)+d(Sx,By)}{2s}\}.$$

Suppose that the pair (A,S) and (B,T) satisfy the (CLR_S) and (CLR_T) properties respectively, then the pair (A,S) and (B,T) have a coincidence point in X. Moreover, if the pair (A,S) and (B,T) are weakly compatible, then A, B, S, T have a unique common fixed point.

Proof: if the pair (A, S) satisfy the CLR_S property, then there exist a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} A_{x_n} = \lim_{n\to\infty} S_{x_n} = p$, where, $p \in S(X)$. As $A(X) \subseteq T(X)$, there exist a sequence $\{y_n\}$ in X such that $A_{x_n} = T_{x_n}$. Hence, $\lim_{n\to\infty} T_{x_n} = p$.

We will show that $\lim_{n\to\infty} B_{x_n} = p$

Putting $x = x_n$ and $y = y_n$ in inequality (1), we arrive at

$$\psi\big(d(Ax_n, By_n)\big) \le \psi\big(M_s(x_n, y_n)\big) - \varphi(M_s(x_n, y_n)),$$

Where

$$M_s(x_n, y_n) = \max\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \tfrac{d(Ax_n, Ty_n) + d(Sx_n, By_n)}{2s}\}.$$

$$= \max\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ay_n), \tfrac{d(Ax_n, Ax_n) + d(Sx_n, By_n)}{2s}\}$$

$$= \max\left\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), \tfrac{d(Sx_n, By_n)}{2s}\right\}.$$

$$\le \max\left\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), s(\tfrac{d(Sx_n, Ax_n) + d(Ax_n, By_n)}{2s}\right\}$$

$$= \max\left\{d(Sx_n, Ty_n), d(Ax_n, Sx_n), d(By_n, Ty_n), (\tfrac{d(Sx_n, Ax_n) + d(Ax_n, By_n)}{2}\right\}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} M_s(x_n, y_n) = \max\{d(p, p), d(p, p), d(By_n, p), \tfrac{d(p, p) + d(p, By_n)}{2s}\}$$

$$= \max\{d(By_n, p), \tfrac{d(p, By_n)}{2}\}. = d(By_n, p),$$

This implies that

$$M_s(x_n, y_n) \le \; d(By_n, p)$$

Making $n \to \infty$ in (2), we get

$$\lim_{n\to\infty} \psi\big(d(Ax_n, By_n)\big) \le \lim_{n\to\infty} \psi\big(M_s(x_n, y_n)\big) - \lim_{n\to\infty} \varphi\big(M_s(x, y)\big)$$

$$\psi\left(\lim_{n\to\infty} d(Ax_n, By_n)\right) \le \psi\left(\lim_{n\to\infty} d(By_n, p)\right) - \varphi\left(\lim_{n\to\infty} d(By_n, p)\right)$$

This implies that, $\lim\limits_{n\to\infty} d(By_n, p) = 0$,

Hence, $\lim\limits_{n\to\infty} By_n = p$.

Now, since the pair (A, S) and (B, T) satisfy the CLR_{ST} property so there exist a sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = p$, where, $p \in S(X) \cap T(X)$, so there exist a point $u \in X$ such that $Su = p$, we have to show that $Au = Su$.

Putting $x = u \; and \; y = y_n$ in equation (1) above, we get,

$$\psi\big(d(Au, By_n)\big) \le \psi\big(M_s(u, y_n)\big) - \varphi(M_s(u, y_n))$$

Where

$$M_s(u, y_n) = \max\{d(Su, Ty_n), d(Au, Sx_n), d(By_n, Ty_n), \tfrac{d(Au, Ty_n) + d(Su, By_n)}{2s}\}$$

$$= \max\{d(u, Ty_n), d(Au, p), d(By_n, Ty_n), \tfrac{d(Au, Ty_n) + d(Su, By_n)}{2s}\}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} M_s(u, y_n) = \max\{d(p, p), d(Au, p), d(p, p), \tfrac{d(Au, p) + d(p, p)}{2s}\}$$

$$= \max\{d(Au, p), \tfrac{d(Au, p)}{2s}\} = d(Au, p)$$

This implies that

$$\psi\big(d(Au,p)\big) \leq \psi\big(d(Au,p)\big) - \varphi(d(Au,p))$$

Hence, $\psi\big(d(Au,p)\big) = 0$.
Therefore, $Au = p = Su$
Thus, u is a point of coincidence of the pair (A,S). As $p \in T(X)$, there exist a point $v \in X$ such that $Tv = p$.
We assert that, $Bv = Tv$
Putting $x = u \; and \; y = v$ in equation (1) above, we get

$$\psi\big(d(Au,Bv)\big) \leq \psi\big(M_s(u,v)\big) - \varphi(M_s(u,v)), \text{ for each } x,y \in X:$$

Where

$$M_s(u,v) = \max\{d(Su,Tv), d(Au,Su), d(Bv,Tv), \tfrac{d(Au,Tv)+d(Su,Bv)}{2s}\}$$

$$= \max\{d(p,p), d(p,p), d(Bv,p), \tfrac{d(p,p)+d(p,Bv)}{2s}\}$$

$$= \max\{d(Bv,p), \tfrac{d(Bu,p)}{2s}\} = d(Bv,p)$$

This implies that

$$\psi\big(d(Bv,p)\big) \leq \psi\big(d(Bv,p)\big) - \varphi(d(Bv,p))$$

Hence, $\psi d(Bv,p) = 0$. This implies that, $Bv = p = Tv$.
Hence, v is a point of coincidence of point (B,T). Since the pair (A,T) is weakly compatible, then $Ap = Sp$.
Now, will show that p is common fixed point of A and S.
Putting $x = p \; and \; y = v$ in equation (1), we get

$$\psi\big(d(Ap,Bv)\big) \leq \psi\big(M_s(p,v)\big) - \varphi(M_s(p,v)), \text{ for each} x,y \in X:$$

Where

$$M_s(p,v) = \max\{d(Sp,Tv), d(Ap,Sp), d(Bv,Tv), \tfrac{d(Ap,Tv)+d(Sp,Bv)}{2s}\}.$$

$$= \max\{d(Ap,P), d(Ap,Ap), d(P,P), \tfrac{d(p,p)+d(Ap,p)}{2s}\}.$$

$$= \max\{d(Ap,p), 2\; \tfrac{d(Bu,p)}{2s}\}$$

$$= d(Ap,p)$$

This implies that

$$\psi\big(d(Ap,p)\big) \leq \psi\big(d(Ap,p)\big) - \varphi(d(Ap,p))$$

Thus, $\psi\big(d(Ap,p)\big) = 0$,
This gives, $Ap = p = Sp$. Hence p is common fixed point of the pair (A,S).
Again let us show that p is common fixed point of B and T.
Putting $x = u \; and \; y = p$ in equation (1), we get
$\psi\big(d(Au,Bp)\big) \leq \psi\big(M_s(u,p)\big) - \varphi(M_s(u.p))$, for each $x,y \in X$:
Where

$$M_s(u,p) = \max\{d(Su,Tp), d(Au,Sp), d(Bu,Tp), \tfrac{d(Au,Tp)+d(Su,Bp)}{2s}\}.$$

$$= \max\{d(p,Tp), d(p,p), d(p,p), \tfrac{d(AuT,p)+d(p,p)}{2s}\}.$$

$$= \max\{d(p,Tp), \tfrac{d(Au,Tp)}{2s}$$

$$= d(p,Tp)$$

This implies that

$$\psi(d(p,Tp)) \leq \psi(d(p,Tp)) - \varphi(d(p,Tp))$$

Thus, $\psi(d(p,Tp)) = 0$,
This gives, $Tp = p = Bp$.
Hence, p is common fixed point of A, B, S, T.
If there exists another fixed point $t \in X$, then from

$$\psi(d(p,t)) = \psi(d(Ap,Bt)) \leq \psi(M_s(p,t)) - \varphi(M_s(p,t)),$$

Where,

$$M_s(p,t) = \max\{d(Sp,Tt), d(Ap,St), d(Bp,Tt), \frac{d(Ap,Tt)+d(Sp,Bt)}{2s}\}.$$

$$= \max\{d(p,t), d(p,p), d(p,p), \frac{d(p,t)+d(p,t)}{2s}\}.$$

$$= \max\{d(p,t), \frac{d(p,p)}{s}\}$$

$$= d(p,t)$$

This implies that

$$\psi(d(p,t)) \leq \psi(d(p,t)) - \varphi(d(p,t))$$

Thus

$$\psi(d(p,t)) = 0.$$

Hence, we conclude that $p = t$.
The proof is completed.
Corollary 2.2 Let (X, d) be a b-metric space and P, T: X→ X be mappings, such that,
$\psi(d(Px,Py)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y))$, for each $x, y \in X$:
Where,

$$M_s(x,y) = \max\{d(Tx,Ty), d(Px,Tx), d(Py,Ty), \frac{d(Px,Ty)+d(Tx,Py)}{2s}\}.$$

Suppose that the pair (P,T) satisfy the $(CLRST)$ properties. Then the pair (P,T) and (B,T) have a common coincidence point in X. Moreover, if the pair $(P,T))$ is weakly compatible, then P, T have a unique common fixed point.

3. Conclusion

In this paper the desired results on the coincidence point for weakly compatible self-mapping satisfying $(\mathrm{CLR_{ST}})$-property obtained and achieve a unique common fixed point for two weakly compatible with the help of altering distance function and ultra-altering distance function in the frame work of b-metric spaces was achieved.

Availability of Data and Materials

On request, the data used to support the findings of this study can be obtained from the corresponding author.

Conflict of Interest

The author declares no conflict of interest.

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